OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2009

FIRST ENGINEERING & INFORMATION TECHNOLOGY EXAMINATION

MATHEMATICS [MA150]

MA153 — *ALGEBRA*

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Time allowed: *Three* hours.
Answer six questions.

1. (a) A small dairy produces three cheeses: mild, standard and mature. The following table summarizes the amount of energy, milk and labour used to produce one box of each of the three cheeses together with the amount of these resources available per day.

Resource	Mild	Standard	Mature	Daily available
Energy	2 kWh	3 kWh	2 kWh	100 kWh
Milk	4 L	4 L	3 L	150 L
Labour	3 h	4 h	6 h	170 h

Solve a system of linear equations to determine the production figures that would ensure all resources are used.

(b) Give one value of t for which the system

$$\begin{aligned} x + 2y &= ty \\ 2x + y &= tx \end{aligned}$$

has infinitely many solutions. (Hint: It might help to rearrange the equations.)

p.t.o.

2. Let

$$A := \begin{pmatrix} 2 & -6 & 0 \\ -2 & -8 & 2 \\ 2 & -8 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 4 & 0 & -3 \\ 1 & 0 & -1 \\ 8 & 1 & -7 \end{pmatrix}.$$

(a) Calculate AB and A^{-1} and then solve

$$\begin{array}{rcrcrcr} 2x & - & 6y & & & = & -4 \\ 2x & + & 8y & - & 2z & = & 2 \\ 2x & - & 8y & & & = & -10 \end{array}.$$

(b) Using $(A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}$ or otherwise, solve

$$\begin{array}{rcrcrcr} 2x & - & 2y & + & 2z & = & 2 \\ -6x & - & 8y & - & 8z & = & 4 \\ & & 2y & & & = & 2 \end{array}.$$

(c) Find a matrix E such that EA is the result of replacing the first row of A by the sum of the first and third rows of A.

3. (a) Calculate the determinant of the matrix

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3+t & 10 \\ 2 & 4 & t+4 \end{pmatrix}$$

and determine all values of t for which A is NOT invertible.

(b) Let $f(t)$ be the area of the parallelogram with vertices $v = (1, 2t)$, $w = (2t, 1)$, $(0, 0)$ and $v + w = (2t + 1, 2t + 1)$. Find a formula for $f(t)$.

(c) Find a 2×2 matrix A such that $f\colon \mathbb{R}^2 \to \mathbb{R}^2, v \mapsto Av$ represents a clockwise rotation through 45° about the origin.

4. (a) Find three vectors $p, v, w \in \mathbb{R}^4$ such that the solutions to the system

$$\begin{array}{rcrcrcrcl} w & + & 3x & + & 3y & + & 2z & = & 1 \\ 2w & + & 6x & + & 9y & + & 5z & = & 5 \\ -w & - & 3x & + & 3y & & & = & 5 \end{array}$$

have the form $p + \lambda v + \mu w$ with $\lambda, \mu \in \mathbb{R}$.

(b) Find all values of k for which the following system has no solution.

$$\begin{array}{rcrcl} x & + & ky & = & 0 \\ kx & + & 9y & = & 1 \end{array}$$

5. (a) Express each of the following in the form $x + iy$ for real numbers x, y:

(i) $\dfrac{1+i}{2+i} \cdot \dfrac{1-2i}{4-3i}$, (ii) $\dfrac{5}{(1-i)(2-i)(3-i)}$, (iii) $\left(\dfrac{-1+\sqrt{3}\,i}{2}\right)^{32}$.

(b) Sketch the Argand diagrams that represent the complex numbers z which satisfy the following conditions:

(i) $|z - 2 + i| = 3$, (ii) $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$, (iii) $\arg(z-2) = \dfrac{\pi}{4}$.

(c) Use de Moivre's Theorem and the Binomial Theorem to find integers A, B, C, D, E and F such that

$$\sin(5\theta) = A\sin^5\theta + B\sin^3\theta + C\sin\theta,$$

$$16\sin^5\theta = D\sin\theta + E\sin(3\theta) + F\sin(5\theta).$$

6. (a) Find all the roots of the polynomial

$$P(z) = z^4 + 6z^3 - 22z^2 - 120z + 400,$$

given that the roots are non-real and that one of the roots is -2 times another root. Hence or otherwise write $P(x)$ as a product of irreducible linear or quadratic factors over $\mathbb{R}$.

(b) Write down the 8th roots of unity in the form $a + ib$, $a, b \in \mathbb{R}$, and illustrate them in the complex plane. State which of the roots are also 4th or 6th roots of unity. Hence, or otherwise write $x^8 - 1$ a product of irreducible linear or quadratic factors over $\mathbb{R}$.

7. (a) Show that the characteristic polynomial $\chi_A(\lambda)$ of the matrix

$$A = \begin{pmatrix} 0 & b & 0 \\ b & a & b \\ 0 & b & a \end{pmatrix}$$

is given by $\chi_A(\lambda) = -\lambda^3 + 2a\lambda^2 + (2b^2 - a^2)\lambda - b^2 a$.

(b) Find the characteristic polynomial, the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 9 & 10 & 0 \\ -9 & -18 & -2 \end{pmatrix}.$$

Hence (or otherwise) find a diagonal matrix D and a matrix E such that $E^{-1}AE = D$.

(c) Show that, if v is an eigenvector for the eigenvalue λ of a matrix M, then λ^2 is an eigenvalue of the matrix M^2, with eigenvector v.

8. (a) Write down the equations of:

(i) an ellipse, (ii) a parabola, (iii) a hyperbola,

in standard form.

(b) Show that

$$4x^2 - 9y^2 - 8x + 72y - 176 = 0$$

represents a hyperbola. Find its foci, vertices, and asymptotes, and sketch its graph.

(c) Find an orthogonal transformation of the axes which reduces the conic

$$xy + x - y = 2$$

to standard form. State what type of conic it is.