History of Topology

Semester I, 2009-10

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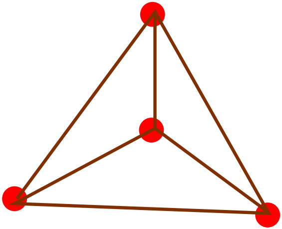
NUI Galway, Ireland

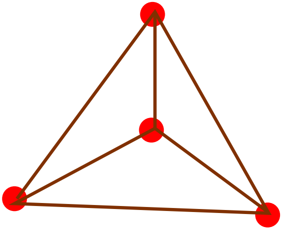
History of Topology Outline

- Lecture 1:
  What is topology?
  Euler circuits in graphs
- Lecture 2:
  Topological invariants:
  Euler-Poincaré characteristic
- Lecture 3:
  One recent application of topology in biology

Tetrahedron

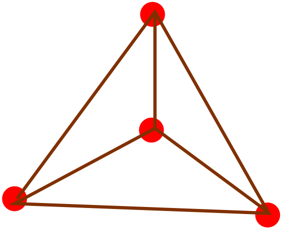


Tetrahedron



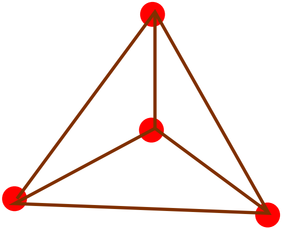
- Number of Vertices?

Tetrahedron



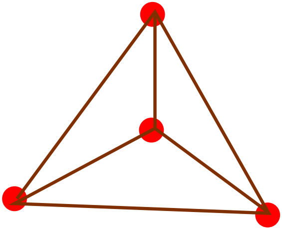
- Number of Vertices?
- Number of edges?

Tetrahedron



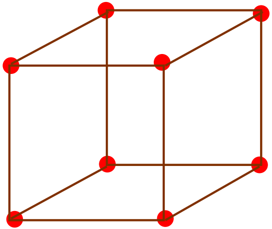
- Number of Vertices?
- Number of edges?
- Number of faces?

Tetrahedron



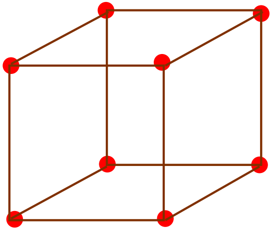
- Number of Vertices? V=4
- Number of edges? E=6
- Number of faces? F=6

Cube



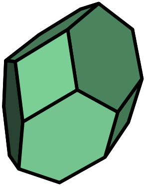
- Number of Vertices? V=
- Number of edges? E=
- Number of faces? F=

Cube



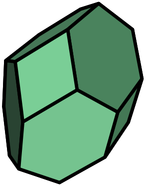
- Number of Vertices? V=8
- Number of edges? E=12
- Number of faces? F=6

Permutahedron



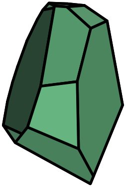
- Number of Vertices? V=
- Number of edges? E=
- Number of faces? F=

Permutahedron



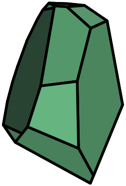
- Number of Vertices? V=24
- Number of edges? E=36
- Number of faces? F=14

Other "polyhedra"



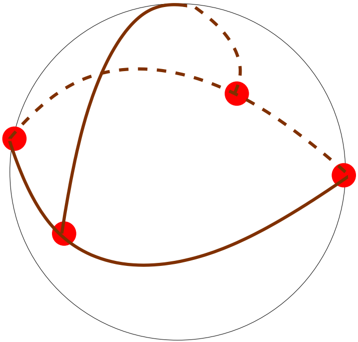
- Number of Vertices? V=
- Number of edges? E=
- Number of faces? F=

Other "surfaces"



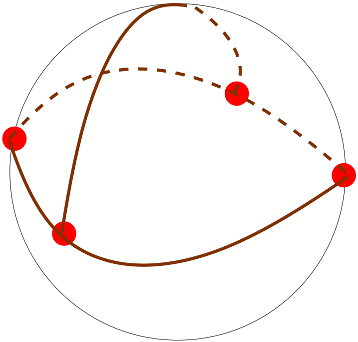
- Number of Vertices? V=30
- Number of edges? E=46
- Number of faces? F=18

Sphere



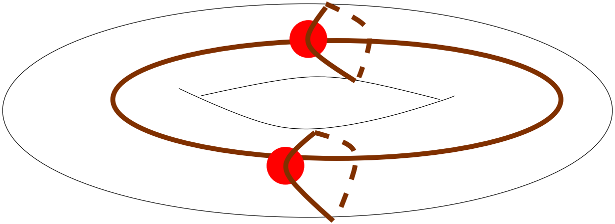
- Number of Vertices? V=
- Number of edges? E=
- Number of faces? F=

Sphere



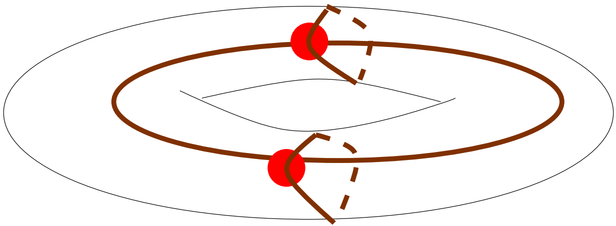
- Number of Vertices? V=4
- Number of edges? E=5
- Number of faces? F=3

Torus



- Number of Vertices? V=
- Number of edges? E=
- Number of faces? F=

Torus



- Number of Vertices? V=2
- Number of edges? E=4
- Number of faces? F=2

Summary

Surface	V	E	F	V-E+F
Tetrahedron	4	6	4	
cube	8	12	6	
Permutahedron	24	36	14	
Other	30	46	18	
Sphere	4	5	3	
Torus	2	4	2	

Summary

Surface	V	E	F	V-E+F
Tetrahedron	4	6	4	2
cube	8	12	6	2
Permutahedron	24	36	14	2
Other	30	46	18	2
Sphere	4	5	3	2
Torus	2	4	2	0

A Scholarly Question

Why is $V - E + F$ so often equal to 2?

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Antoine-Jean Lhuilier published an article in 1813 in which he observes that Euler's formula is wrong for surfaces (like the torus) of solids with holes in them. He showed that $V - E + F = 2 - 2g$ where g is the number of holes in the solid.

A modern answer

Rather than look at Euler's proof, let's look at a modern explanation (due to W. Thurston at Princeton) of why $V - E + F$ equals 2 for so many of our examples.

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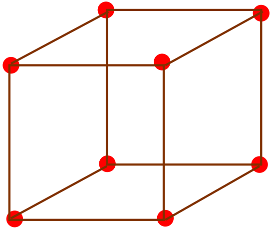
Let P be any one of our examples other than the torus. (For instance, P could be the cube.)

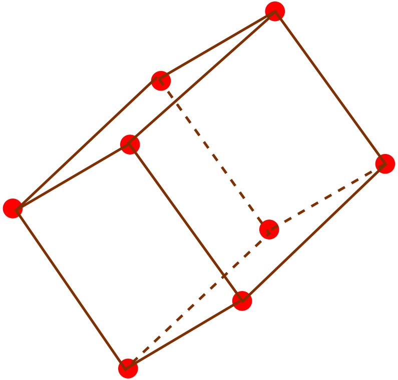


A modern answer (cont)

Rotate the surface P so that no edge is horizontal.

A modern answer (cont)

Rotate the surface P so that no edge is horizontal. For P equal to the cube we could rotate as follows.



A modern answer (cont)

Now place a "+" on each vertex and on each face. Place a "-" on each edge.

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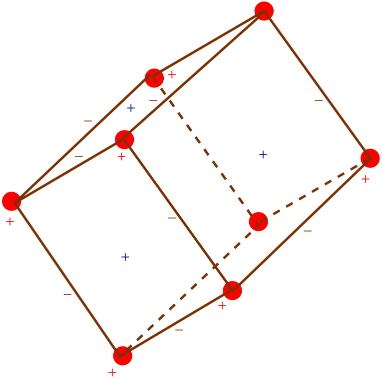

A modern answer (cont)

Imagine an axis through the top and bottom vertices of P, with a gentle wind swirling in one direction around the axis.

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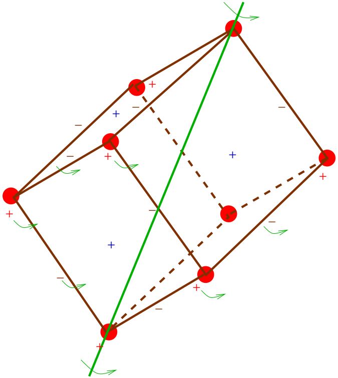

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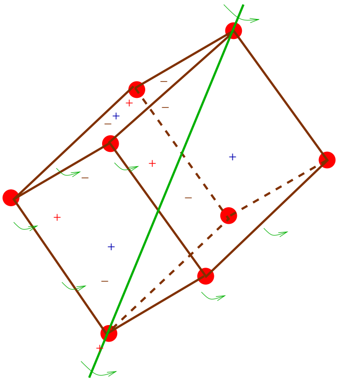
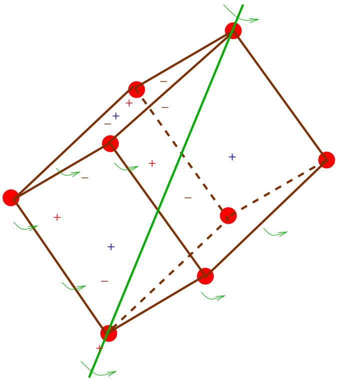


A modern answer (cont)

Now let all the vertex "+"s and edge "-"s blow into the neighbouring face. For P equal to the cube we have the following.



In any face there are as many "+"s as "-"s. The top and bottom vertex "+"s don't blow into any face. So $V - E + F = 2$.

A hidden assumption

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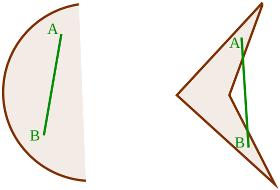

Convex Not convex

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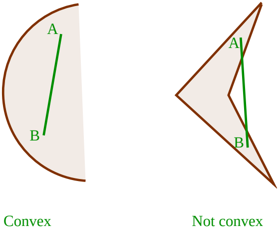

Euler only proved his formula for surfaces of convex bodies.

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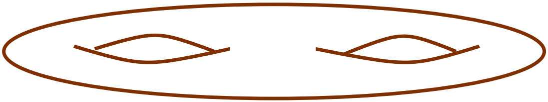
Consider, for example, the surface of a double doughnut



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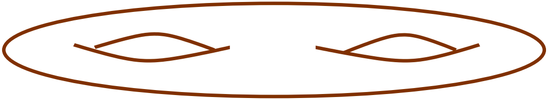

Not matter how we divide this surface into polygonal regions, we'll have $V - E + F = -2$

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A rigorous proof of the higher-dimensional formula was not obtained until a decade or so later, and depends on the notion of *homotopy*.

Homology theory and homotopy theory are major areas of current international mathematical research.