History of Topology

Semester I, 2009-10

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History of Topology Outline

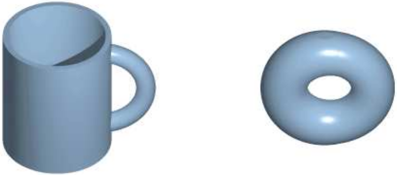
- Lecture 1:
  What is topology?
  Euler circuits in graphs
- Lecture 2:
  Topological invariants:
  Euler-Poincaré characteristic
- Lecture 3:
  One recent application of topology in biology

What is Topology?

Topology is the study of those properties of an object that remain unchanged throughout a continuous deformation of the object.

Coffee mugs and doughnuts

A topologist can't tell the difference between a coffee mug and a doughnut!



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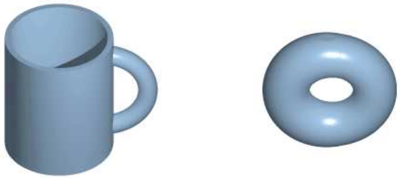

Click to see how a doughnut can be continuously deformed into a coffee mug.

Coffee mugs and doughnuts

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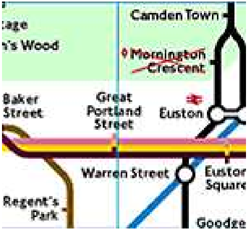
Click to see how a doughnut can be continuously deformed into a coffee mug.

A topological property of a mug/doughnut is, for example, that it has precisely one 1-dimensional hole

Topological properties in every-day life



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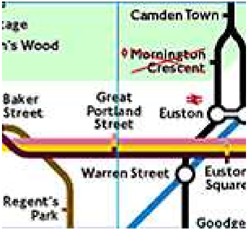

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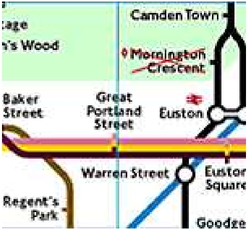

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This illustartes that distance is NOT a topological property.

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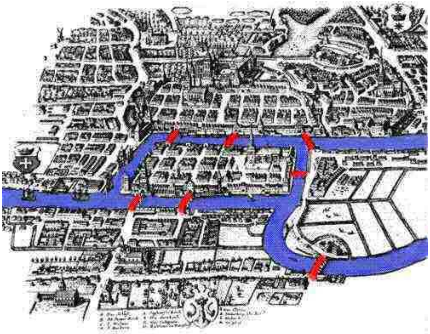
(The solution of a problem relating to the geometry of position)

Euler and the bridges of Königsberg



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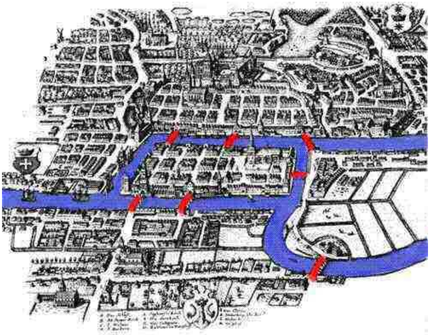
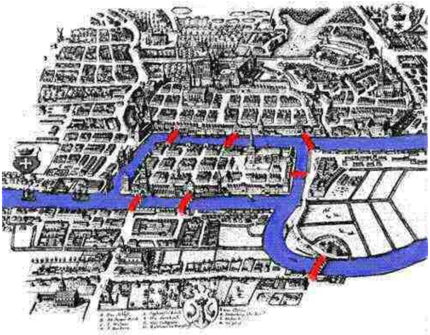


Problem:

Euler wanted to know if he could organize a pub-crawl that crossed each of the seven bridges precisely once.

Euler and the bridges of Königsberg

The problem involved a different type of geometry where distance was not relevant.



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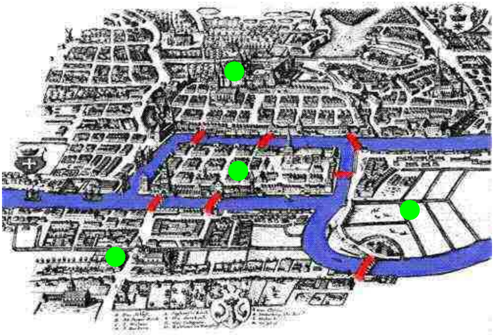

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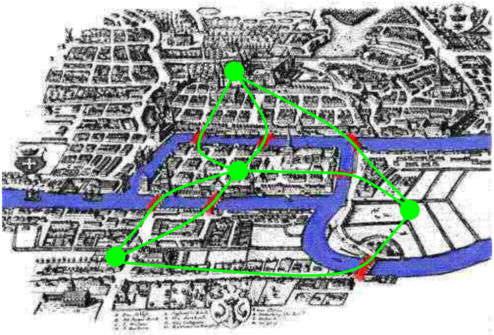
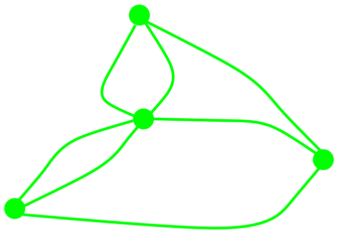


To tackle the problem Euler placed a vertex in each land mass.

And inserted an edge between vertices for each bridge connecting the corresponding land masses.

A graph problem

Euler's pub-crawl puzzle becomes a scholarly question about the following graph.



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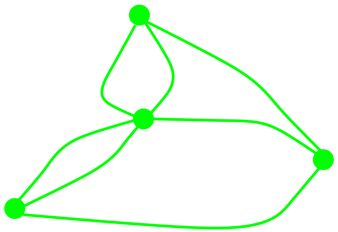


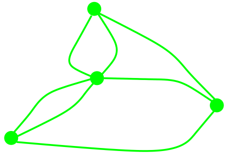
Scholarly question:

Does the graph have a path that traverses each edge exaclty once?

A scholarly answer: The first topological theorem

Definition:

The *degree* of a vertex in a graph is the number of edges incident with it.



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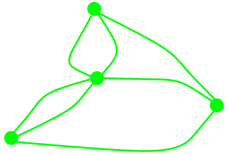

In 1736 Euler proved the following.

Theorem:

A **connected** graph has a path traversing each edge exactly once if, and only if, exactly zero or two vertices have odd degree.

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A graph G is connected if any two vertices are joined by a path.

One half of Euler's theorem

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If the initial and final vertices are the same then this initial/final vertex must have even degree.

Otherwise the initial and final vertex each have odd degree.

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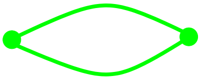
We use induction on the number n of edges in the graph. The proposition is certainly true for $n = 1$ and $n = 2$.



Inductive hypothesis: Assume the proposition holds for all graphs with fewer than n edges where $n \geq 3$.

The other half of Euler's theorem
(Difficult material for enthusiasts. Not on the exam!)

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- Consider the graph G' consisting of all vertices in G and all those edges in G not in P.
- If G' contains any edge then it must contain a connected subgraph in which each vertex has even degree. (Why?)

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- By the inductive hypothesis this connected subgraph contains an Euler circuit.

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- Consider the graph G' consisting of all vertices in G and all those edges in G not in P.
- If G' contains any edge then it must contain a connected subgraph in which each vertex has even degree. (Why?)
- By the inductive hypothesis this connected subgraph contains an Euler circuit.
- This Euler circuit could be added to P to make P longer. But the length of P is maximal, so in fact G' contains no edge and P is an Euler trail of G.