First test: Wednesday 21 February

Example The spaces

$(-1, 1)$ and $\mathbb{R}$

are homeomorphic because of the homeomorphism

$$f : (-1,1) \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x}{1-x^2}.$$

Let aim towards showing that the following four spaces are distinct (non-homeomorphic):

$[-1, 1]$

$\mathbb{R}$

$\mathbb{R}^2$

S^1

Recall A property is topological if, whenever X has the property, then so too does any Y homeomorphic to X.

Proposition Let $f : X \to Y$ be a homeomorphism. If X is connected then so too is Y.

Proof Suppose Y is not connected. Then there exist open sets $U, V \subseteq Y$ such that $Y = U \cup V$ and $U \cap V = \phi$. Since f is continuous

$$f^{-1}(U) = \{x \in X : f(x) \in U\}$$

is open in X. So too is

$$f^{-1}(V).$$

Since f is a homeomorphism (and hence surjective)

$$f^{-1}(U) \cup f^{-1}(V) = X$$

Also

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset$$

Hence X is not connected.

$\boxtimes$

<u>Example</u> Let's show that $\mathbb{R}^2$ is <u>not</u> homeomorphic to $\mathbb{R}$.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ were a homeomorphism.

$X = \mathbb{R}^2 \setminus \{(1,2)\}$

$Y = \mathbb{R} \setminus \{f(1,2)\}$

Exercise: If $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism then so too is $f : X \to Y$.

But X is connected and Y is <u>not</u> connected.

Hence there can be no homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}$.

Towards compactness

We'd like to say that $[-1, 1]$ is "finite" and that $(-\infty, \infty)$ is "infinite".

We'll introduce a new concept:

Compactness