MA342: Tutorial Problems 2014-15

Tutorials: Tuesday, 1-2pm, Venue = AC214
Wednesday, 2-3pm, Venue = AC201

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PROBLEMS

# 1 Euler characteristics

1. Draw a graph on a sphere $\mathbb{S}^2$



in such a way that if two edges intersect then they intersect in a vertex of the graph. Determine the number of vertices $V$, edges $E$ and faces $F$ for your graph. Then compute the Euler characteristic $\chi(\mathbb{S}^2) = V - E + F$.

2. Prove that the value of the Euler characteristic $\chi(\mathbb{S}^2) = V - E + F$ in Problem 1 does not depend on your particular choice of graph on the sphere. [See Lecture 1.]

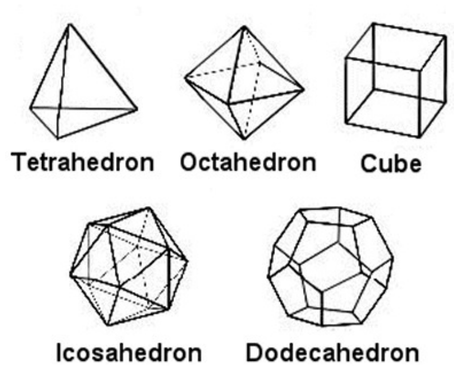
3. A *platonic solid* is a 3-dimensional convex object whose surface is the union of a finite number of polygonal planar faces satisfying:

   (a) all faces are congruent to some fixed regular $p$-gon;

   (b) the intersection of two faces is either empty or a common edge of the two faces or a common vertex of the two faces;

   (c) the same number of faces, $q$, meet at each vertex.

   Five platonic solids are shown in the following figure.



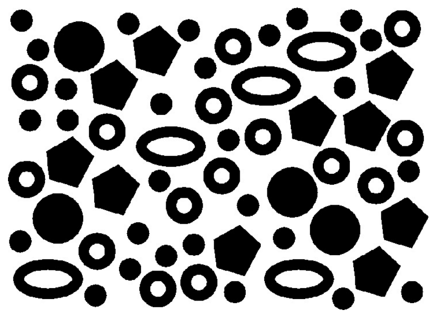
Use the Euler characteristic to prove

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}\,, \qquad e \geq 0$$

for any platonic solid.

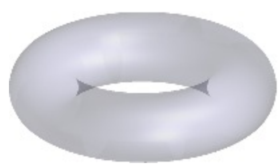
4. Deduce from Problem 3 that there are only five platonic solids.

5. The digital image



represents a region $X \subseteq \mathbb{R}^2$ formed as a union of various unit squares $[m, m+1] \times [n, n+1]$ for various integers $m, n$. Determine $\chi(X)$.

6. Draw a graph on a torus $\mathbb{T}$



in such a way that

   (a) if two edges of the graph intersect then they intersect in a vertex of the graph;

   (b) each resulting face on the torus is a curvilinear disk (*i.e.* a "continuous deformation" of some planar polygonal disk).

   Determine the number of vertices $V$, edges $E$ and faces $F$ for your graph. Then compute the Euler characteristic $\chi(\mathbb{T}) = V - E + F$. [The term "continuous deformation" will be made precise later in the course: it is just a *homeomorphism*.]

7. Prove that the value of the Euler characteristic $\chi(\mathbb{T}) = V - E + F$ in Problem 6 does not depend on your particular choice of graph on the torus. [Hint: The torus $\mathbb{T}$ can be constructed from a rectangular sheet of paper by identifying/gluing opposite sides of the sheet. We know that the Euler characteristic of a solid plane recangle is 1.]

8. A *polygonal surface* is a union of curvilinear polygonal disks such that, if two polygonal disks intersect, then their intersection is a union of edges and/or vertices of the disks. The polygonal disks are called *faces*. The soccer ball is an example of a

polygonal surface with pentagonal and hexagonal faces. For any polygonal surface $X$ we define the *Euler characteristic* $\chi(X) = V - E + F$ where $V$ is the number of vertices on $X$, $E$ the number of edges and $F$ the number of faces.

Suppose that $X$ is a polygonal surface. Let $A, B \subseteq X$ be subsets of $X$ each arising as a union of faces. Prove that

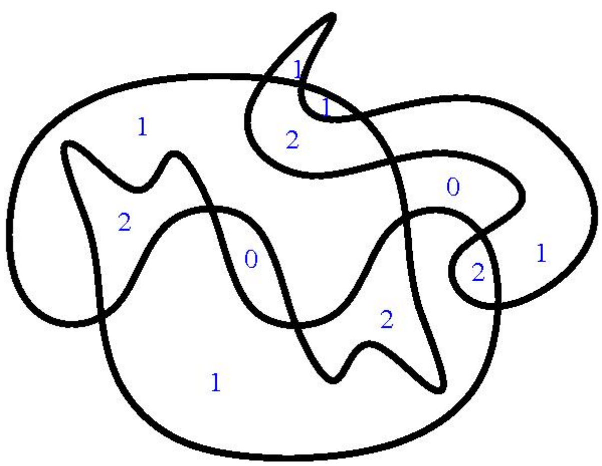
$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

[See Lecture 2.]

9. Use the formula $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ to determine the Euler characteristic of a double torus surface.

# 2 Euler Integration

1. The following picture shows the boundaries of several regions $U_1, \ldots, U_t \subseteq \mathbb{R}^2$ of common Euler characteristic $\chi(U_i) = 1$.




No two boundaries are tangential at any point. The numbers in the interiors of the regions and their intersections represent the weight function $\omega\colon X \to \mathbb{N}$ where $X = U_1 \cup U_2 \cup \ldots \cup U_t$ and $w(x) = |\{i : x \in U_i\}|$. Evaluate the Euler integral

$$\int_X \omega \, d\chi$$

and then determine the number of regions $t$.

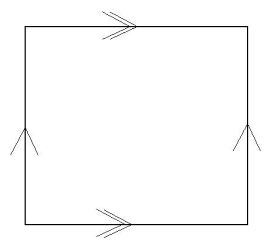
2. Let $X \subseteq \mathbb{R}^2$ be a region arising as the union of subregions $U_1, U_2, \ldots U_t \subseteq X$ of common Euler characteristic $\chi(U_i) = C$. Let $\omega\colon X \to \mathbb{N}$ be the weight function given by $w(x) = |\{i : x \in U_i\}|$. Prove that

$$t = \frac{1}{C} \int_X \omega \, d\chi \ .$$

[See Lecture 4.]

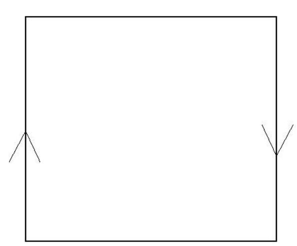
## 3 Möbius strips, Klein bottles ...

1. The torus $\mathbb{T}$ is obtained from the unit square $[0,1] \times [0,1]$ by making the identifications $(x,0) = (x,1)$ and $(0,y) = (1,y)$ for $x, y \in [0,1]$.



Is it true that any loop in $\mathbb{T}$ that has no self intersections cuts $\mathbb{T}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{T}$ into two components. [See Lecture 4]

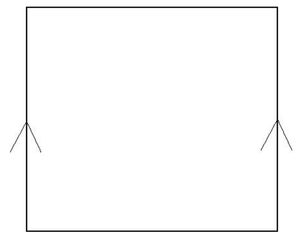
2. The Möbius strip $\mathbb{M}$ is obtained from the unit square $[0,1] \times [0,1]$ by making the identifications $(0,y) = (1, 1-y)$ for $y \in [0,1]$.



Is it true that any loop in $\mathbb{M}$ that has no self intersections cuts $\mathbb{M}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{M}$ into two components. [See Lecture 4]

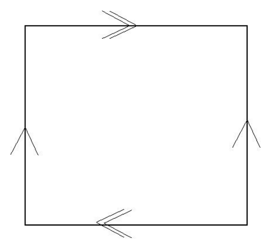
3. The cylinder $X$ is obtained from the unit square $[0,1] \times [0,1]$ by making the identi-

fications $(0, y) = (1, y)$ for $y \in [0, 1]$.



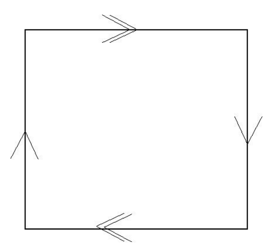
Is it true that any loop in $X$ that has no self intersections cuts $X$ into two components? If not, then exhibit a loop that does not cut $X$ into two components.

4. The Klein bottle $\mathbb{K}$ is obtained from the unit square $[0, 1] \times [0, 1]$ by making the identifications $(x, 0) = (1 - x, 1)$ and $(0, y) = (1, y)$ for $x, y \in [0, 1]$.



Is it true that any loop in $\mathbb{K}$ that has no self intersections cuts $\mathbb{K}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{K}$ into two components.

5. The projective plane $\mathbb{P}$ is obtained from the unit square $[0, 1] \times [0, 1]$ by making the identifications $(x, 0) = (1 - x, 1)$ and $(0, y) = (1, 1 - y)$ for $x, y \in [0, 1]$.



Is it true that any loop in $\mathbb{P}$ that has no self intersections cuts $\mathbb{P}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{P}$ into two components.

# 4 Subsets of Euclidean space

1. Exhibit a collection of open subsets of the plane $\mathbb{E}^2$ whose intersection is not open. [Lecture 5]

2. Let $X \subset \mathbb{E}^2$ be the set of those points in the plane that have at least one rational coordinate. Is $X$ an open subset of $\mathbb{E}^2$? Is $X$ a connected subset of $\mathbb{E}^2$? Justify your answers.

3. Let

$$Y = \{(0, y) \in \mathbb{E}^2 \,:\, -1 < y < 1\}\,,$$
$$Z = \{(x, \sin(1/x) \in \mathbb{E}^2 \,:\, 0 < x \leq 1\}\,,$$
$$X = Y \cup Z\,.$$

Is $X$ an open subset of $\mathbb{E}^2$? Is $X$ a connected subset of $\mathbb{E}^2$? Justify your answers.

# 5 Topological spaces

1. For each of the following sets $X$ and collections $T$ of open subsets decide if the pair $X, T$ satisfies the axioms of a topological space. If it does, determine whether $X$ is connected. If it is not a topological space then explain which axioms fail.

   (a) $X = \mathbb{R}^n$ and the subset $U \subset X$ is open if, for any $x \in U$, there is a real $\epsilon > 0$ such that the open Euclidean ball $B^n(x, \epsilon)$ of radius $\epsilon$ and centred at $x$ is contained in $U$.

   (b) $X = \mathbb{R}^n$ and the subset $U \subset X$ is open if, for any $x \in X \setminus U$, there is a real $\epsilon > 0$ such that the open Euclidean ball $B^n(x, \epsilon)$ of radius $\epsilon$ and centred at $x$ is contained in the complement $X \setminus U$.

   (c) $X = \mathbb{R}^n$ and every subset $U \subset X$ is open.

   (d) $X = \mathbb{R}^n$ and the only open subsets are $X$ and the empty set $\emptyset$.

   (e) $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if its complement $\mathbb{Z} \setminus U$ is finite or $U = \emptyset$.

   (f) $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if $U$ is finite or $U = \mathbb{Z}$.

   (g) $X = \mathbb{R}^n$ and a subset $U \subset \mathbb{R}^n$ is open if and only if it is a vector subspace of $\mathbb{R}^n$. Here $\mathbb{R}^n$ has the standard addition and scalar multiplication.

   (h) $X = \{1, 2, 3, 4\}$ and $T = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$.

   (i) $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if each of its elements is even.

# 6 Subspaces

1. Given a topological space $X$, define what it means for a subset $Y \subseteq X$ to be a *subspace*.

2. For each of the following topological spaces $X$ and subspaces $Y \subseteq X$ describe the connected components of $Y$.

   (a) $X = \mathbb{E}^2$, $Y = \{(x, y) \in \mathbb{E}^2 \,:\, x^2 + y^2 \neq 1\}$.

   (b) $X = \mathbb{E}^1$, $Y = \mathbb{Q}$.

(c) $X = \mathbb{E}^2$, $Y = \{(x,y) \in \mathbb{E}^2 \,:\, x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$.

(d) $X = \mathbb{E}^2$, $Y = \{(x,y) \in \mathbb{E}^2 \,:\, x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$.

3. The table

| | $H$ | $M$ | $R$ | $C$ | $W$ |
|---|---|---|---|---|---|
| $H$ | 0 | 11 | 10 | 14 | 22 |
| $M$ | 11 | 0 | 3 | 13 | 21 |
| $R$ | 10 | 3 | 0 | 12 | 20 |
| $C$ | 14 | 13 | 12 | 0 | 16 |
| $W$ | 22 | 21 | 20 | 16 | 0 |

gives distances between the species Human, Mouse, Rat, Cat, Whale. For $\epsilon > 0$ let $G_\epsilon$ denote the graph with vertices $H, M, R, C, W$ and with an edge between vertices $X$ and $Y$ if $dist(X,Y) \le \epsilon$.

(a) Sketch the graphs $G_4$, $G_{10}$, $G_{16}$.

(b) Explain how one can view the graphs $G_\epsilon$ as subspaces of $\mathbb{E}^5$.

(c) Draw the dendrogram that describes the inclusion relationships between the connected components of the subspaces $G_0, G_2, G_4, \ldots, G_{18}, G_{20}$.

# 7 Some useful jargon

1. Let $X$ be a topological space and let $W \subseteq X$ be some subset.

- The subset $W$ is said to be *closed* if the complement $X \setminus W$ is an open subset of $X$.
- A point $x \in X$ is said to be a *limit point* of $W$ if every open set $U \subset X$ containing $x$ has non-empty intersection with $W$.
- The union of $W$ and all its limit points in $X$ is said to be the *closure* of $W$. The closure is denoted by $\overline{W}$.

(a) Prove that the closure $\overline{W}$ is a closed subset of $X$.

(b) Suppose that $W \subseteq Z$ where $Z$ is a closed subset of $X$. Prove that $\overline{W} \subseteq Z$.

(c) Prove that $\overline{W}$ is equal to the intersection of all closed subsets of $X$ containing $W$.

(d) Prove that a subset $W$ is closed if and only if $W = \overline{W}$.

2. Find a family of closed subsets of the real line whose union is not closed.

3. Describe the closure of the subspace $W = \{(1/n)\sin(n) \,:\, n = 1, 2, \ldots\}$ of the real line.

4. Let $Y$ be a subspace of $X$. Show that if $A$ is closed in $Y$ and if $Y$ is closed in $X$ then $A$ is closed in $X$.

# 8 Continuity

1. Give the definition of a continuous function between topological spaces.
2. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions. Prove that the composite $gf\colon X \to Z, x \mapsto g(f(x))$ is continuous.
3. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are homeomorphisms. Prove that the composite $gf\colon X \to Z, x \mapsto g(f(x))$ is a homeomorphism.

The first in-class test will consist of a few of the above questions.

4. Prove that the unit circle $S^1 = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 = 1\}$ is homeomorphic to the square $Y = \{(x, y) \in \mathbb{E}^2 : -2 \leq x, y \leq 2, \text{ and either } x \in \{-2, 2\} \text{ or } y \in \{-2, 2\}\}$.
5. Let $\Delta$ denote an equilateral triangular region in the plane $\mathbb{E}^2$. Describe the construction of a continuous surjective function $f\colon [0, 1] \to \Delta$. (You are *not* asked to prove any convergence nor to prove surjectivity.)
6. Prove that if $X$ and $Y$ are homeomorphic then $X$ is connected if and only if $Y$ is connected.
7. Prove that $(0, 1)$ is homeomorphic to $\mathbb{E}$.
8. Prove that $\mathbb{E}$ is not homeomorphic to $\mathbb{E}^2$.
9. Prove that $\mathbb{E}$ is not homeomorphic to the space $Y = \{(x, y) \in \mathbb{E}^2 : x = 0 \text{ or } y = 0\}$. (Here $Y$ is the union of the $x$-axis and $y$-axis.)

# 9 Compactness

1. Explain why $\mathbb{E}^2$ is not compact.
2. Prove that the interval $[0, 1]$ is compact.
3. Prove that if $X$ is compact and $f\colon X \to Y$ is a continuous map then the image of $f$ is compact.
4. Prove that if $X$ is compact and $Y$ is homeomorphic to $X$ then $Y$ is compact.
5. Determine the accumulation points of the subset $A = \{1/n\}_{n=1,2,3,\ldots}$ of $\mathbb{R}$.
6. Determine the accumulation points of the subset $A = (0, 1)$ of $\mathbb{R}$.
7. Prove that a subset $A$ of a topological space $X$ is closed if and only if it contains all its accumulation points.

8. Describe a surjective continuous map $f\colon [0,1] \to \Delta$ where $\Delta \subset \mathbb{E}^2$ is a solid equilateral triangle. Explain why $f$ is surjective, stating clearly any theorems that you use in your explanation.

9. Prove that a compact subset $A$ of a Hausdorff space $X$ is closed.

# 10 Simplicial complexes

1. Describe a triangulation on the torus. Determine the number of $k$-simplices in your triangulation for $k = 0, 1, 2$ and then compute the Euler characteristic.

2. Determine the Euler characteristic of the Möbius band.

3. Determine the Euler characteristic of the sphere $S^n = \{x \in \mathbb{E}^{n+1} : ||x|| = 1\}$ for $n = 1, 2, 3, ....$

4. Describe a triangulation on the double torus.

   Determine the number of $k$-simplices in your triangulation for $k = 0, 1, 2$ and then compute the Euler characteristic.

The second in-class test will consist of a few of the above questions. Questions covered by the first test might also appear on the second test.

# 11 Homotopy of maps

1. Let $Y \subset \mathbb{E}^n$ be an arbitrary convex subset of Euclidean space and let $X$ be an arbirary topological space. Prove that any two continuous maps $f, g\colon X \to Y$ are homotopy equivalent.

2. Prove that homotopy equivalence of maps $f \simeq g$ is an equivalence relation on the set of continuous maps $X \to Y$ from a given space $X$ to a given space $Y$.

3. Let $f\colon [0,1] \to S^1$ be a continuous map. Prove that there is a unique continuous map $\tilde{f}\colon [0,1] \to \mathbb{R}$ such that $f(t) = e(\tilde{f}(t))$ where $e\colon \mathbb{R} \to S^1, \theta \mapsto e^{2\pi\theta\mathbf{i}}$.

4. Define the *winding number* of a map $f\colon S^1 \to S^1$ with. Explain why homotopic maps $f \simeq g$ have the same winding number.

5. Let $[S^1, S^1]$ denote the set of homotopy classes of maps $S^1 \to S^1$. Describe a bijection $\omega\colon [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$. Explain why $\omega$ is onto. Explain why $\omega$ is injective.

6. State and prove the Fundamental Theorem of Algebra.

## 12 Homotopy equivalent spaces

1. Prove that any convext subspace $Y \subset \mathbb{E}^n$ is homotopy equivalent to the space consisting of a single point.

2. Prove that the complex plane minus the origin $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle $S^1$.

3. Prove that homotopy equivalence of maps $f \simeq g$ is an equivalence relation on the set of continuous maps $X \to Y$ from a given space $X$ to a given space $Y$.

4. Use the fact that the Euler characteristic of a triangulable space is a homotopy invariant to prover Brouwer's fixed point theorem: any continuous map $D^n \to D^n$ on the closed disc has at least one fixed point.

5. Prove the Frobenius-Perron Theorem: a real square matrix with positive entries has a positive real eigenvalue and the corresponding eigenvector has positive components.

## 13 Nash Equilibrium

1. Describe what is meant by a *Nash Equilibrium*, explaining any concepts from Game Theory that you use.

2. Use Brouwer's fixed point theorem to prove the existence of a Nash equilibrium in a mixed strategy game.