Example $X = \mathbb{R}$, usual topology.

Let $\mathcal{F} = \{(n-2, n+2)\}_{n \in \mathbb{Z}}$.

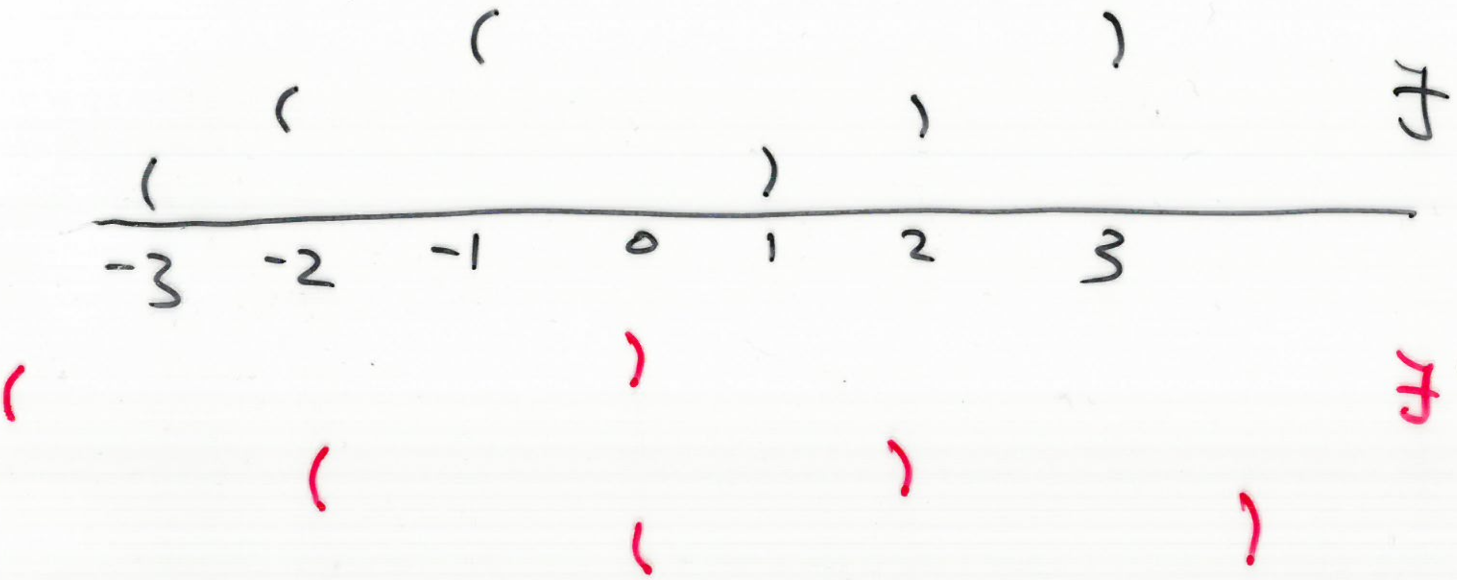
Then $\mathcal{F}$ is an open cover of $\mathbb{R}$.

Example (continued)

Let $2\mathbb{Z}$ be the even integers.

Then

$$\mathcal{F}' = \{(n-2, n+2)\}_{n \in 2\mathbb{Z}}$$



This $\mathcal{F}'$ is a subcover of $\mathcal{F}$ since the union of $\mathcal{F}'$ is $X = \mathbb{R}$.

Defn A topological space X is <u>compact</u> if every open cover of X has a finite subcover.

Example $\mathbb{R}$, with standard topology, is <u>not</u> compact.

Consider $\mathcal{F} = \{(n-2, n+2)\}_{n \in \mathbb{Z}}$.

The following proves that compactness is a topological property.

Proposition Suppose $f: X \to Y$ is a continuous map. Suppose X is compact. If $f: X \to Y$ is surjective then Y is also compact.

Proof Assume X is compact and f is surjective.

Let $\mathcal{F}$ be an open cover of Y. Then $\{f^{-1}(U)\}_{U \in \mathcal{F}} = \mathcal{G}$ is an open cover of X. (Think!)

Since X is compact there is a finite subcover $\mathcal{G}'$ of $\mathcal{G}$.

Let say

$$\mathcal{G}' = \{f^{-1}(U)\}_{U \in \mathcal{F}'} .$$

But then

$$\{U\}_{U \in \mathcal{F}'}$$

is a finite subcover of $\mathcal{F}$.

$\blacksquare$

How could we show that $[0,1]$ is <u>not</u> homeomorphic to $\mathbb{R}$? The following does the trick.

<u>Theorem</u> $[0,1]$ is compact.

<u>Proof</u> Let $\mathcal{F}$ be an open cover of $[0,1]$.

Define

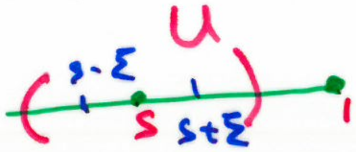
$$X = \{ x \in [0,1] : [0,x] \text{ is contained in the union of a finite subfamily of } \mathcal{F} \}.$$

- X is non-empty since $0 \in X$.
- if $0 \leq y \leq x$ and if $x \in X$ then $y \in X$.

- X has a least upper bound s since X is bounded above by 1. (Completeness Axiom for $\mathbb{R}$.)

We need to show that $s = 1$ and $s \in X$, as that will mean that a finite subfamily of $\mathcal{F}$ has union equal to $[0,1]$.

Suppose $s \neq 1$. Let $U \in \mathcal{F}$ be chosen with $s \in U$.



Choose $\varepsilon > 0$ with $(s-\varepsilon, s+\varepsilon) \subseteq U$. Let $\mathcal{F}'$ be a finite subfamily of $\mathcal{F}$ whose union contains $[0, s-\varepsilon]$. Then $\mathcal{F}' \cup \{U\}$ has union containing $[0, s+\varepsilon]$. But then s is not the least upper bound of X. Hence $s = 1$.

Let $U \in \mathcal{F}$ be chosen with $s = 1 \in U$. Choose $\varepsilon > 0$ with

the open set

$$(1-\varepsilon, 1] = [0,1] \cap (1-\varepsilon, 1+\varepsilon)$$

contained in $\mathcal{U}$. Let* $\mathcal{F}'$ be a finite subfamily of $\mathcal{F}$ whose union contains $[0, 1-\frac{\varepsilon}{2}]$. Then the union of the finite family $\mathcal{F}' \cup \{\mathcal{U}\}$ contains $[0,1]$. So $s = 1 \in X$.

*we can do this since $1-\frac{\varepsilon}{2} < s$.

$\boxed{\mathcal{U}}$