Recap

$f: X \to Y$ is continuous if $f^{-1}(U)$ is open in X for all open $U \subset Y$.

X is homeomorphic to Y is there is a continuous $f: X \to Y$ and a continuous $g: Y \to X$ with $f \circ g = 1$, $g \circ f = 1$. We call f a homeomorphism.

observe: if f is a homeomorphism then clearly f is injective and surjective.

Example Consider

$$f: [0, 2\pi) \longrightarrow S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$$

$$\theta \longmapsto (\cos\theta, \sin\theta)$$

Note that f is continuous, injective and surjective, but f is not a homeomorphism

Topological Properties

A property is said to be <u>topological</u> if, whenever some space X has this property, then so too do all space Y homeomorphic to X.

<u>Example</u>

"Being finite" is a topological property but it is not one that interests topologists too much.

<u>Example</u> "furthest Euclidean distance between two points" is not a topological property.

Example The following proposition shows that "connectedness" is a topological property.

Proposition Let $f : X \to Y$ be a homeomorphism. If X is connected then so too is Y.

Proof Suppose Y is **not** connected. Then there exist open subsets $U, V \subset Y$ such that U, V are both non-empty, and $Y = U \cup V$, and $U \cap V = \emptyset$. Since f is continuous

$$f^{-1}(U) = \{x \in X : f(x) \in U\}$$

is open in X.

Now

$$f^{-1}(U) \cup f^{-1}(V) = X$$

clearly.

Also

$$f^{-1}(U) \cap f^{-1}(V) = \phi$$

clearly.

So X is not connected.

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<u>Theorem</u> The space $[0, 2\pi)$ is not homeomorphic to S^1.

<u>proof</u>

suppose there were a homeomorphism

$$f : [0, 2\pi) \longrightarrow S^1 .$$

Consider

$$X = [0, 2\pi) \setminus \{1\} = [0, 1) \cup (1, 2\pi)$$

$$Y = S^1 \setminus \{f(1)\}$$

Exercise: The function

$$f': X \to Y, \quad x \longmapsto f(x)$$

is a homeomorphism.

But X is not connected and Y is connected, so f' can't be a homeo^{m}. So f can't be a homeo^{m}.

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Towards Compactness

We'd like to say that $[0,1]$ is "finite" and that $(-\infty, \infty)$ is "infinite". But we can't, so we'll introduce a new word: compact.

Let X be a topological space. Let $\mathcal{F}$ be a family of open subsets of X whose union equals X. We say that $\mathcal{F}$ is an <u>open cover</u> of X.

If $\mathcal{F}'$ is a subfamily of $\mathcal{F}$, and if the union of all sets in $\mathcal{F}'$ equals X, then we say that $\mathcal{F}'$ is a <u>subcover</u> of $\mathcal{F}$.

An open cover is said to be <u>finite</u> if it involves just finitely many sets.

Definition

A topological space X is compact if every open cover $\mathcal{F}$ of X has a finite subcover $\mathcal{F}'$.