Continuity

Defn Let X, Y be topological spaces. A function $f: X \to Y$ is continuous if the inverse image

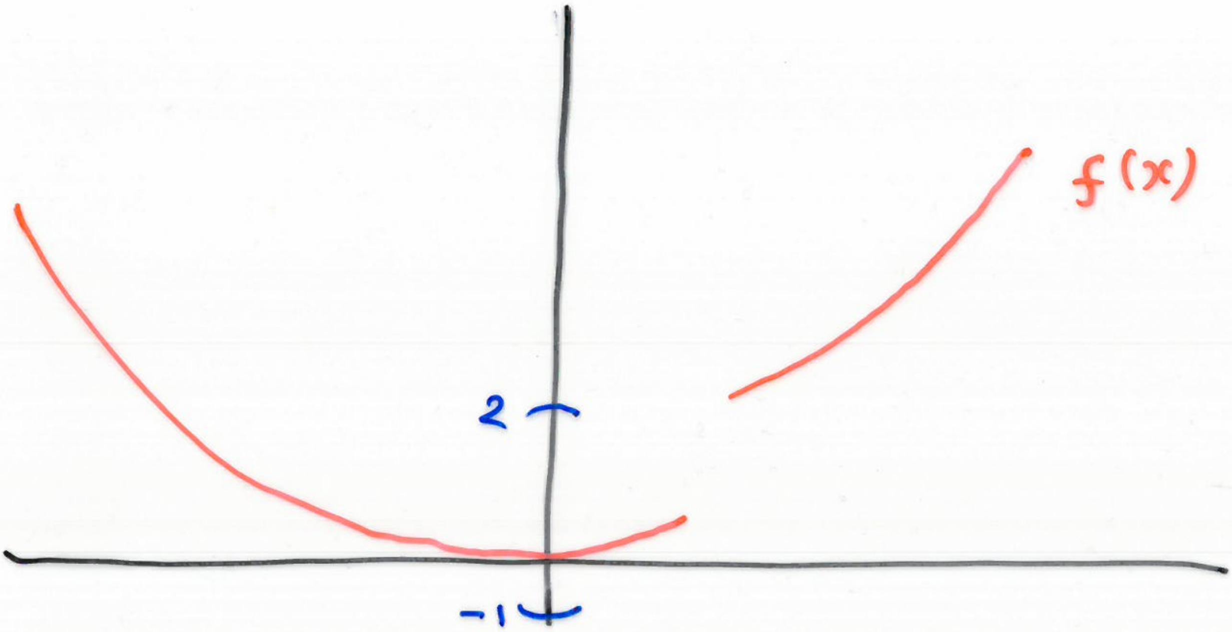
$$f^{-1}(U) = \{ x \in X : f(x) \in U \}$$

of every open set $U \subseteq Y$ is open in X

Example $X = \mathbb{E}^1$, $Y = \mathbb{E}^1$.

Consider $f : \mathbb{E}^1 \to \mathbb{E}^1$ given by

$$f(x) = \begin{cases} x^2 & , \quad x \leq 1 \\ x^2+1 & , \quad x > 1 \end{cases}$$



Consider the open subset

$$U = (-1, 2) \subseteq \mathbb{E}^1 = Y$$

The pre-image

$f^{-1}(U) = (-\sqrt{2}, 1]$ is <u>not</u> open in $\mathbb{E}^1 = X$.

Hence f is <u>not</u> continuous.

<u>Example</u> $X = (-\infty, 1) \cup (1, \infty)$

$Y = \mathbb{R}$

Define $g : X \to Y$ by

$$g(x) = \begin{cases} x^2, & x \geq 1 \\ x^2+1, & x < 1 \end{cases}$$

This function g is continuous.

Major Definition: A continuous function of topological spaces $f : X \to Y$ is a homeomorphism if there exists a continuous function $g : Y \to X$ such that

$$g(f(x)) = x \quad \text{for all } x \in X$$

and

$$f(g(y)) = y \quad \text{for all } y \in Y.$$

Defn Two spaces X, Y are homeomorphic if there exists some homeomorphism $f : X \to Y$.

<u>Example</u> The unit circle

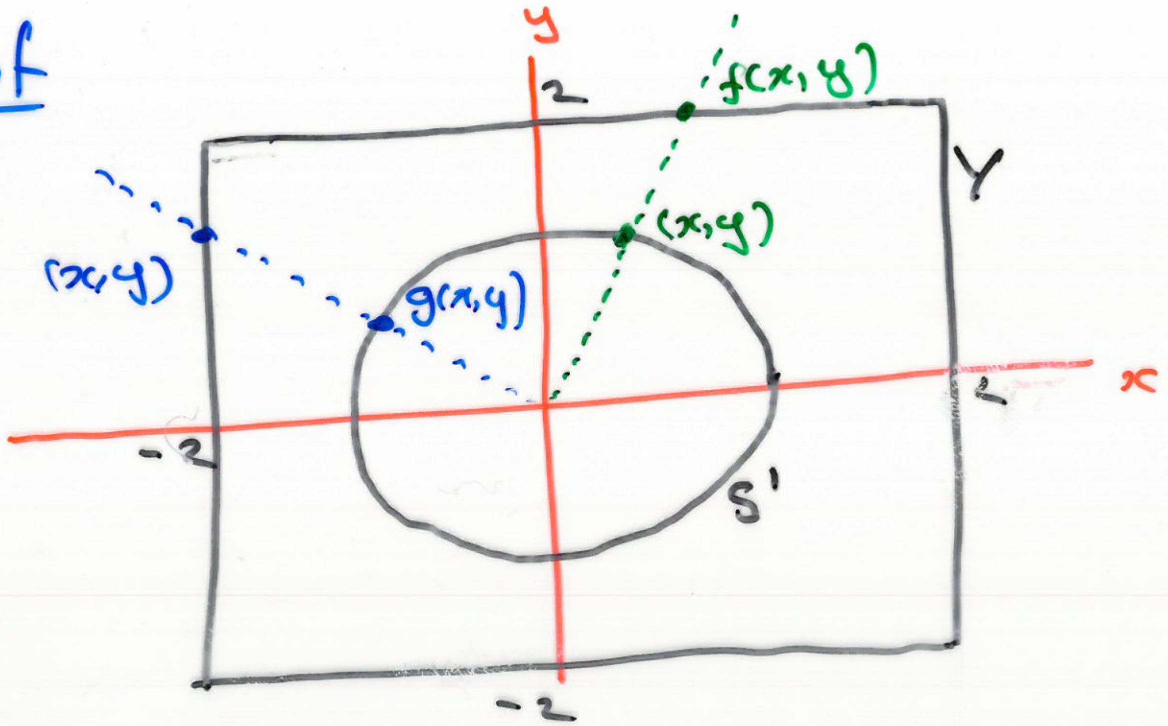
$$S^1 = \{ (x,y) \in \mathbb{E}^2 : x^2+y^2 = 1 \}$$

is homeomorphic to the square

Y of side 4.

$$Y = \{ (x,y) \in \mathbb{E}^2 : -2 \leq x, y \leq 2 \text{ and either } x \in \{-2, 2\} \text{ or } y \in \{-2,2\} \}$$

<u>Proof</u>



Consider $f : S^1 \to Y$, $(x,y) \longmapsto f(x,y)$

where $f(x,y)$ is the point of

intersection of the ray from the origin through point (x, y) and the space Y.

Consider $g : Y \to S^1$, $(x,y) \mapsto g(x,y)$ where $g(x,y)$ is the intersection of the ray through the ~~origi~~ ~~and~~ point (x,y) from the origin and the space $X = S^1$.

Note that both f and g are continuous, and that

$$g(f(x,y)) = (x,y)$$

and

$$f(g(x,y)) = (x,y).$$

Hence the square and circle are homeomorphic. $\square$

Proposition if $f : X \to Y$ and $h : Y \to Z$ are continuous functions of topological spaces, Then their composite

$$h \circ f : X \to Z, \quad x \mapsto h(f(x))$$

is continuous.

Proof Let $U \subset Z$ be any open set in Z. Then $h^{-1}(U) \subset Y$ is open in Y because h is continuous.

But $f^{-1}(h^{-1}(U)) \subset X$ is open in X because f is continuous, and $h^{-1}(U)$ is open in Y.

Now

$$(h \circ f)^{-1}(U) = f^{-1}(h^{-1}(U)).$$

So $h \circ f$ is continuous because the preimage of any open set in Z is open in X.

$\boxed{M}$