Example Let $X = \mathbb{R}^n$. Let $\mathcal{T}$ consist of all subsets of X. Then $(X, \mathcal{T})$ is a topological space. We call this the discrete topology.

This space is not connected.

For instance

$$U = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$$

$$V = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}.$$

Then $U \cup V = X$, $U \cap V = \emptyset$,

$U \neq \emptyset \neq V$.

Example Let $X = \mathbb{R}^n$. Let $\mathcal{T}$ consist of just two open sets

$$\mathcal{T} = \{\emptyset, X\}.$$

Then $(X, \mathcal{T})$ is a topological space. We call $\mathcal{T}$ the trivial topology.

This topological space is connected.

<u>Definition</u> Let X be a set with a topology T.
Let $Y \subseteq X$ be a subset of X.
In the <u>subspace topology</u> on Y a subset $U \subseteq Y$ if and only if

$$U = Y \cap A$$

with A an open subset of X.
With this topology we call Y a <u>topological subspace</u> of X.

<u>Example</u> Let $X = \mathbb{R}$ with the standard topology in which a set $U \subseteq \mathbb{R}$ is open if, for each $x \in U$, there is an $\varepsilon > 0$ with

$$(x - \varepsilon, x + \varepsilon) \subseteq U .$$

Consider the integers $\mathbb{Z} \subset \mathbb{R}$, with subspace topology. Then the space $\mathbb{Z}$ is <u>not</u> connected because

$$U = \{ n \in \mathbb{Z} : n \geq 0 \}$$

$$V = \{ n \in \mathbb{Z} : n < 0 \}$$

are open in the subspace, $\mathbb{Z} = U \cup V$, $U \cap V = \emptyset$, $U \neq \emptyset \neq V$.

$$U = \mathbb{Z} \cap (-\tfrac{1}{2}, \infty)$$

$$V = \mathbb{Z} \cap (-\infty, -\tfrac{1}{2})$$

<u>Definition</u> A <u>connected component</u> of a topological space X is a connected subspace $Y \subseteq X$ such that there is no connected subspace $W \subseteq X$ with $Y \subsetneq W$.

Example

Let $X = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 \neq 1\}$

There are two connected components of X, namely

$$Y = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 < 1\},$$

and

$$Z = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 > 1\}.$$

$Y =$ 

$Z =$