Definition (Riesz [1909], Hausdorff [1914])

A **topological space** consists of a set X and a collection T of subsets of X which we call **open**. The following axioms must hold.

T1) The union of any collection of open sets is open.

T2) The intersection of any **finite** collection of open sets is open

T3) Both $\emptyset$ and X are open.

Example 1

$X = \{1, 2, 3, 4\}$

$T = \{\emptyset, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}.$

This is a topological space.

Example 2

$X = \{1, 2, 3, 4\}$

$T = \{\phi, \{1,2\}, \{2,3\}, \{1,2,3\},$

$\{1,2,3,4\}\}.$

Not a topological space. T2 fails.

Example 3

$X = \{1, 2, 3, 4\}$

$T = \{\phi, \{2\}, \{3\}, \{1,2,3\}, \{1,2,3,4\}\}$

Not a topological space. T1 fails.

Example 4

$X = \{1,2,3,4\}$

$T = \{\phi, \{1\}, \{2\}, \{1,2\}, \{1,2,3\},$

$\{1,2,3,4\}\}.$

This is a topological space.

__Example 5__ Let $X = \mathbb{Z}$. The <u>cofinite</u> <u>topology</u> on X has, as open sets, those subsets $U \in X$ such that the complement $X \setminus U$ is finite. Also, the empty set $\emptyset$ is deemed to be open. This is a topological space.

For the next example we need some notation.

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we define the <u>Euclidean</u> <u>norm</u>

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

For $x, y \in \mathbb{R}^n$ we define the <u>Euclidean</u> <u>distance</u>

$$d(x,y) = \|x - y\|.$$

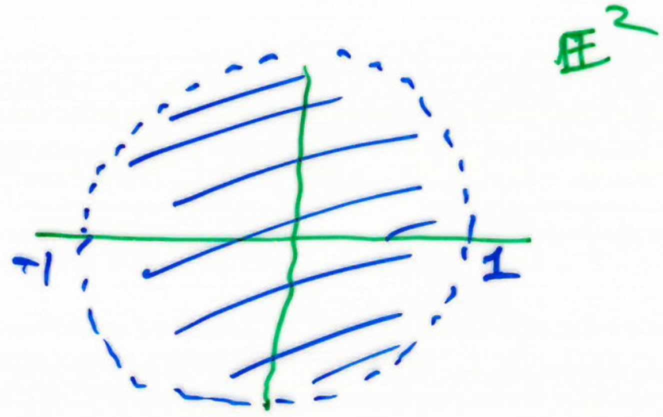
we write $\mathbb{E}^n$ to denote the set $\mathbb{R}^n$ endowed with the Euclidean distance.

For $x \in \mathbb{E}^n$ and for any real number $\varepsilon > 0$ we define the open ball centred at x of radius ε to be

$$B^n(x, \varepsilon) = \{ y \in \mathbb{E}^n : d(x,y) < \varepsilon \}$$

Examples

$B^2((0,0), 1)$



$B^1(0, 1)$

Example 6 Let $X = \mathbb{R}^n$.

Let T consist of those subsets $U \subseteq \mathbb{R}^n$ such that, for any $x \in U$, we can find an $\varepsilon > 0$ such that the Euclidean ball

$$B^n(x, \varepsilon)$$

lies entirely in U:

$$B^n(x, \varepsilon) \subseteq U.$$

This defines a topological space.

Definition A topological space X is said to be **connected** if it can **not** be expressed as a union

$$X = U \cup V$$

where U, V are non-empty open subsets of X, and where $U \cap V = \emptyset$.

Example 1 is connected

Examples 2 & 3 not topological spaces

Example 4 is connected.

Example 5 $\mathbb{Z}$ is connected with the cofinite topology

Example 6 $\mathbb{E}^n$ is connected.