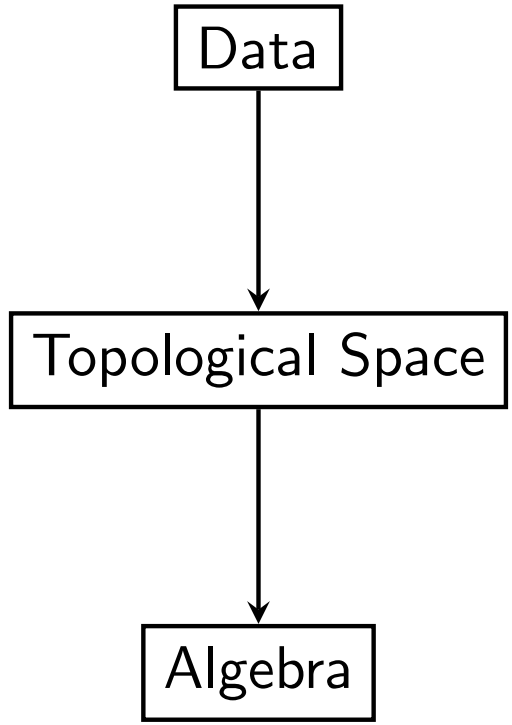
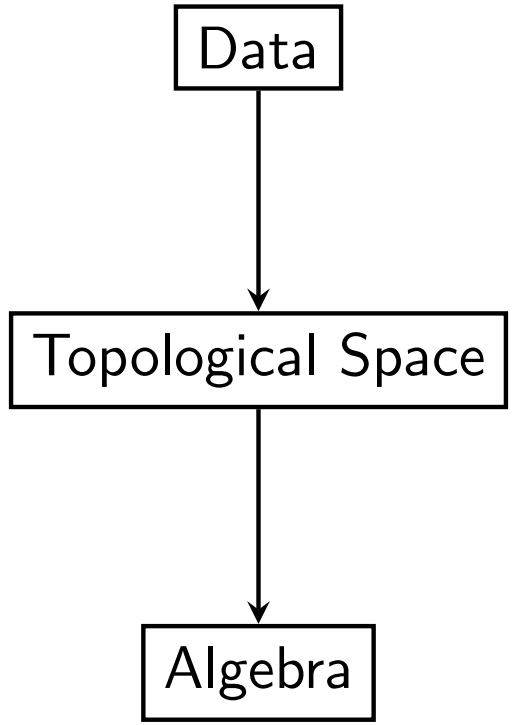
A flavour of topological data analysis

Final MA342 lecture


Data
Topological Space
Algebra



General Aim: Given a finite sample S from an unknown population X we'd like to use algebra to describe/construct a topological space BS that models X.

Cluster Analysis

Consider toy data points $S = \{v_1, v_2, \ldots, v_{72}\} \subset \mathbb{R}^{262144}$

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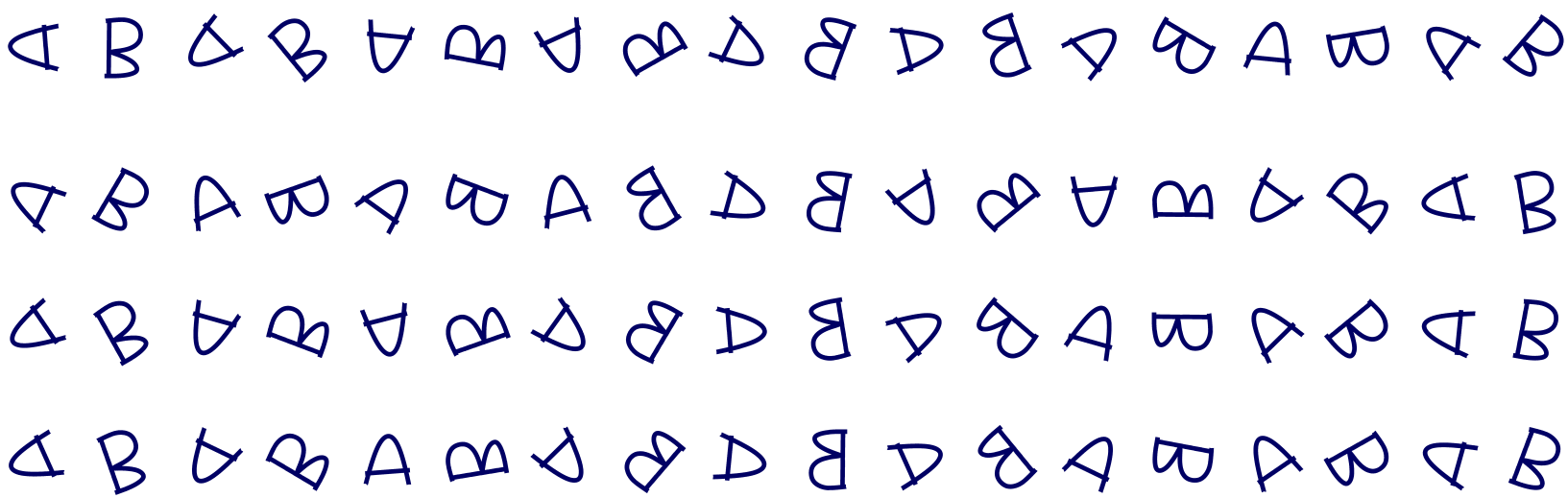

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- n-simplices the subsets $\sigma \subseteq S$ with $n+1$ vertices and $d(v, v') \leq \epsilon_t$ for all $v, v' \in \sigma$.

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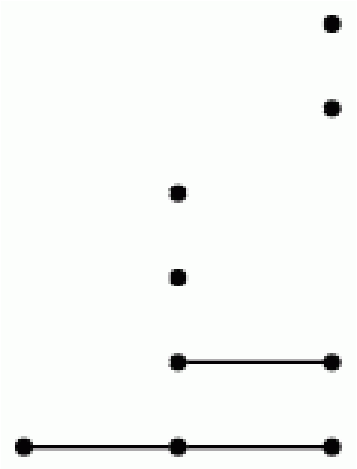
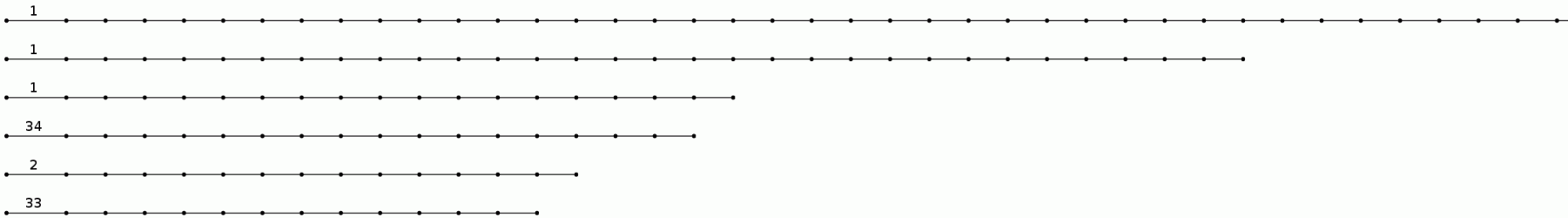
A β_n bar code has

$\beta_n^{s,t}$ *horizontal lines from column s to column t*

$$(\beta_2^{s\,t}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$



β_0 barcode for the toy data S



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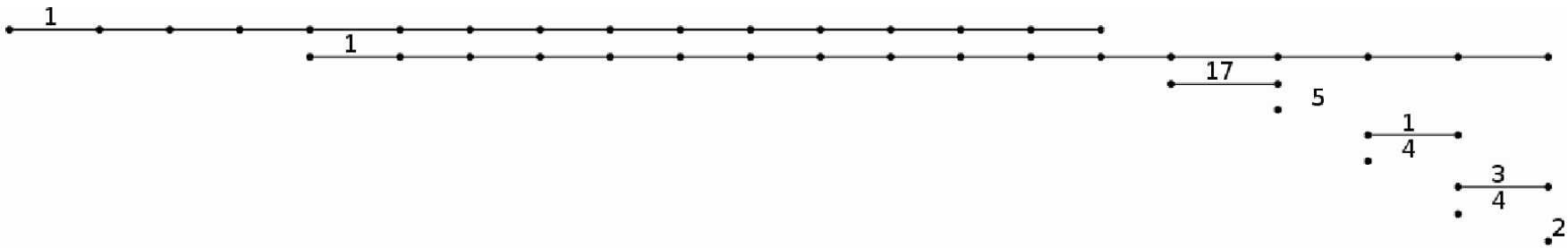
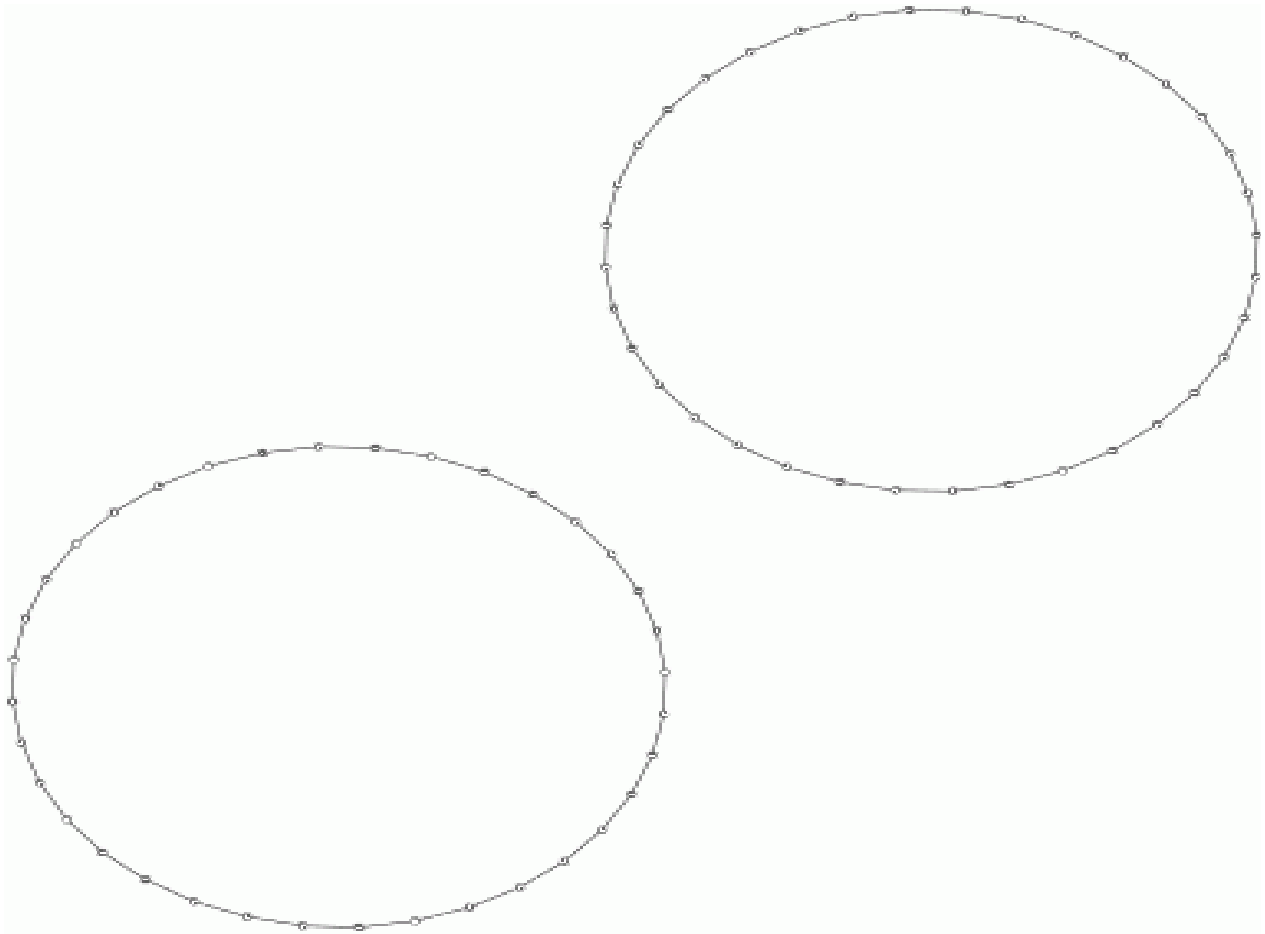
$$H_n(Y, \mathbb{F}) = \ker(\mathbb{F}^{s_n} \xrightarrow{\partial_n} \mathbb{F}^{s_{n-1}}) / \mathrm{im}(\mathbb{F}^{s_{n+1}} \xrightarrow{\partial_{n+1}} \mathbb{F}^{s_n})$$

$s_n =$ number of n-simplices in Y

β_1 barcode for the toy data S



Data Model: A homotopy retract $Y \subset Y_{20}$



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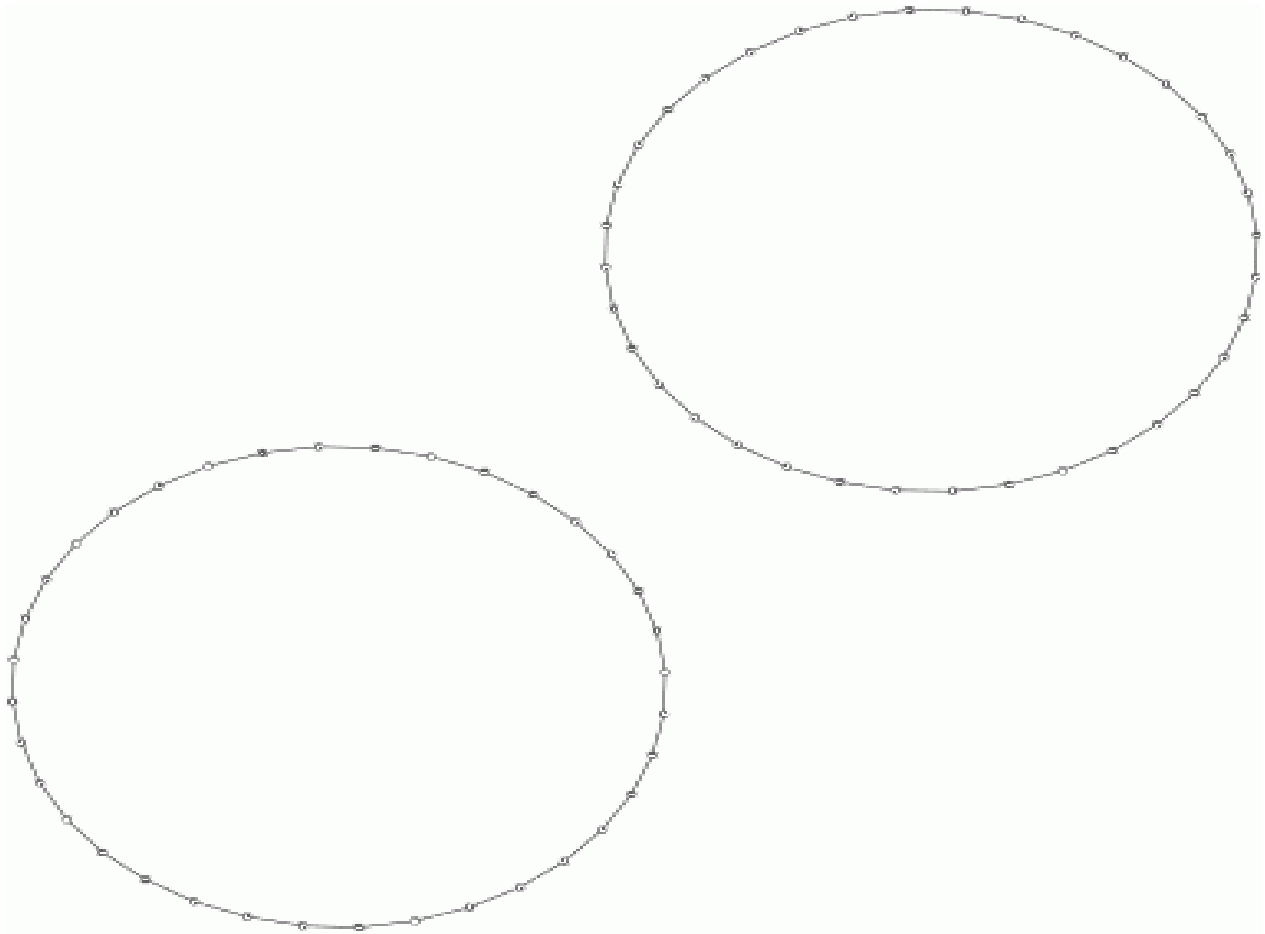

$$Y \simeq S^1 \sqcup S^1$$

Basic Mapper Cluster Analysis

G. Singh, F. Mémoli & G. Carlsson (2007)

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Distances $d_X(v, v')$ for $v, v' \in S \subset X$, X an unknown metric space.

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Simplicial nerve of $\mathcal{V}$.

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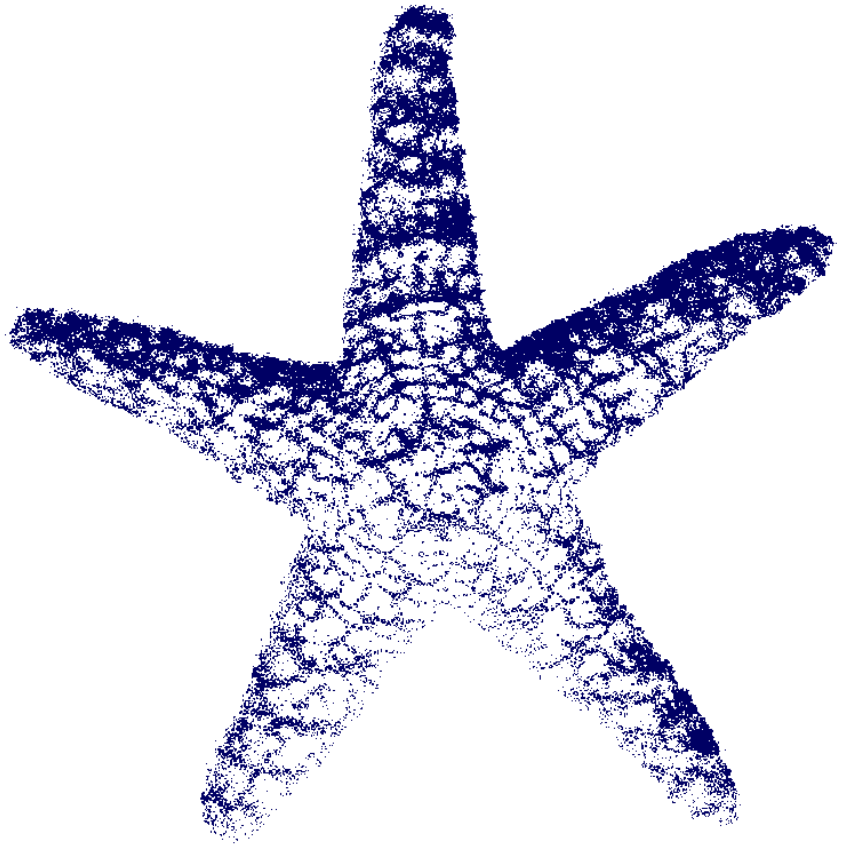

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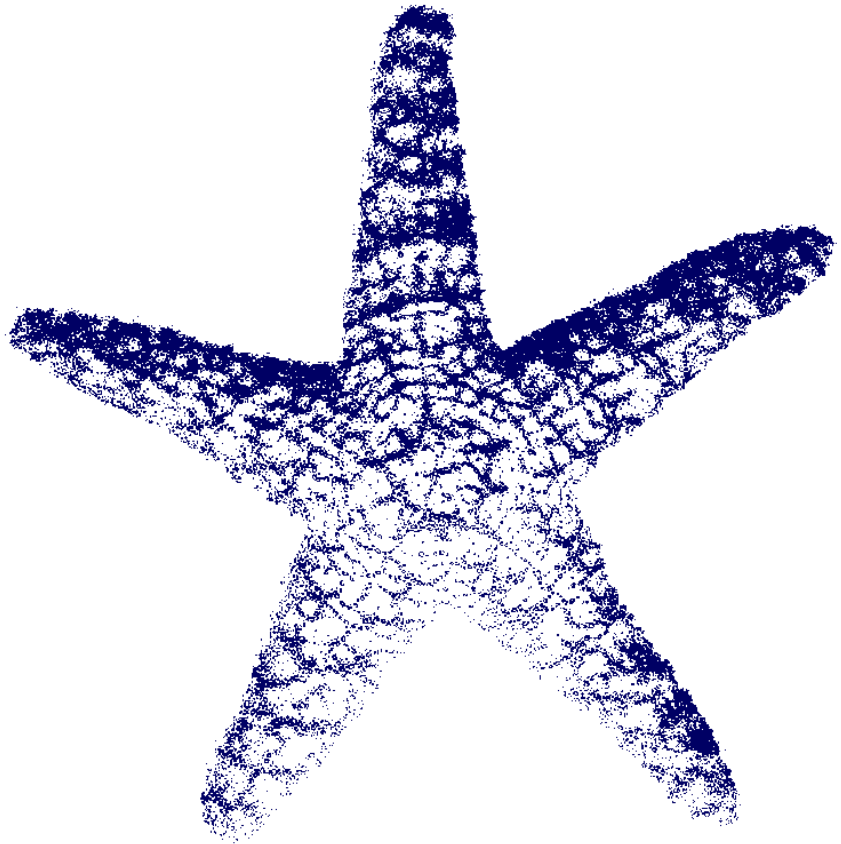
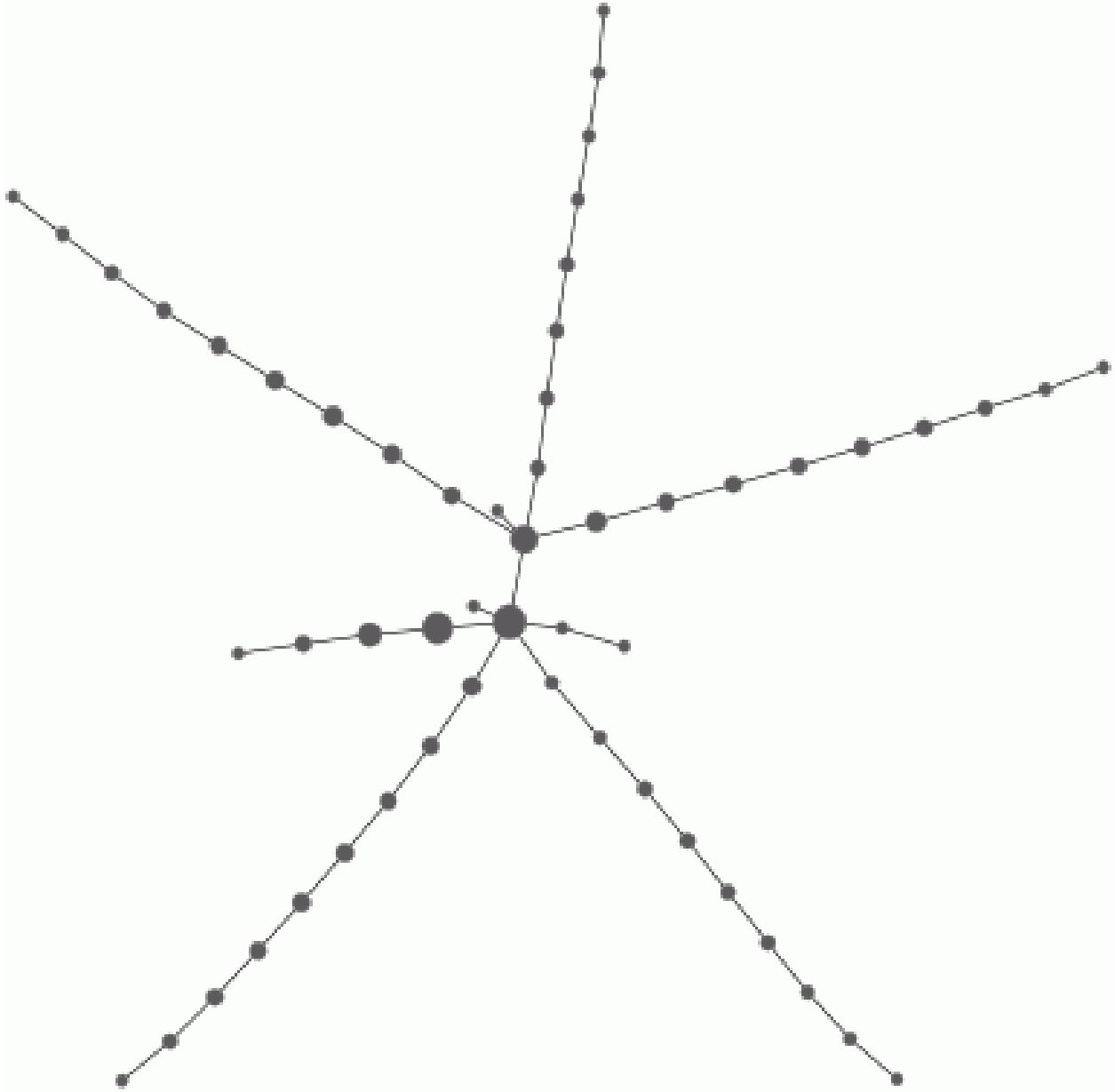


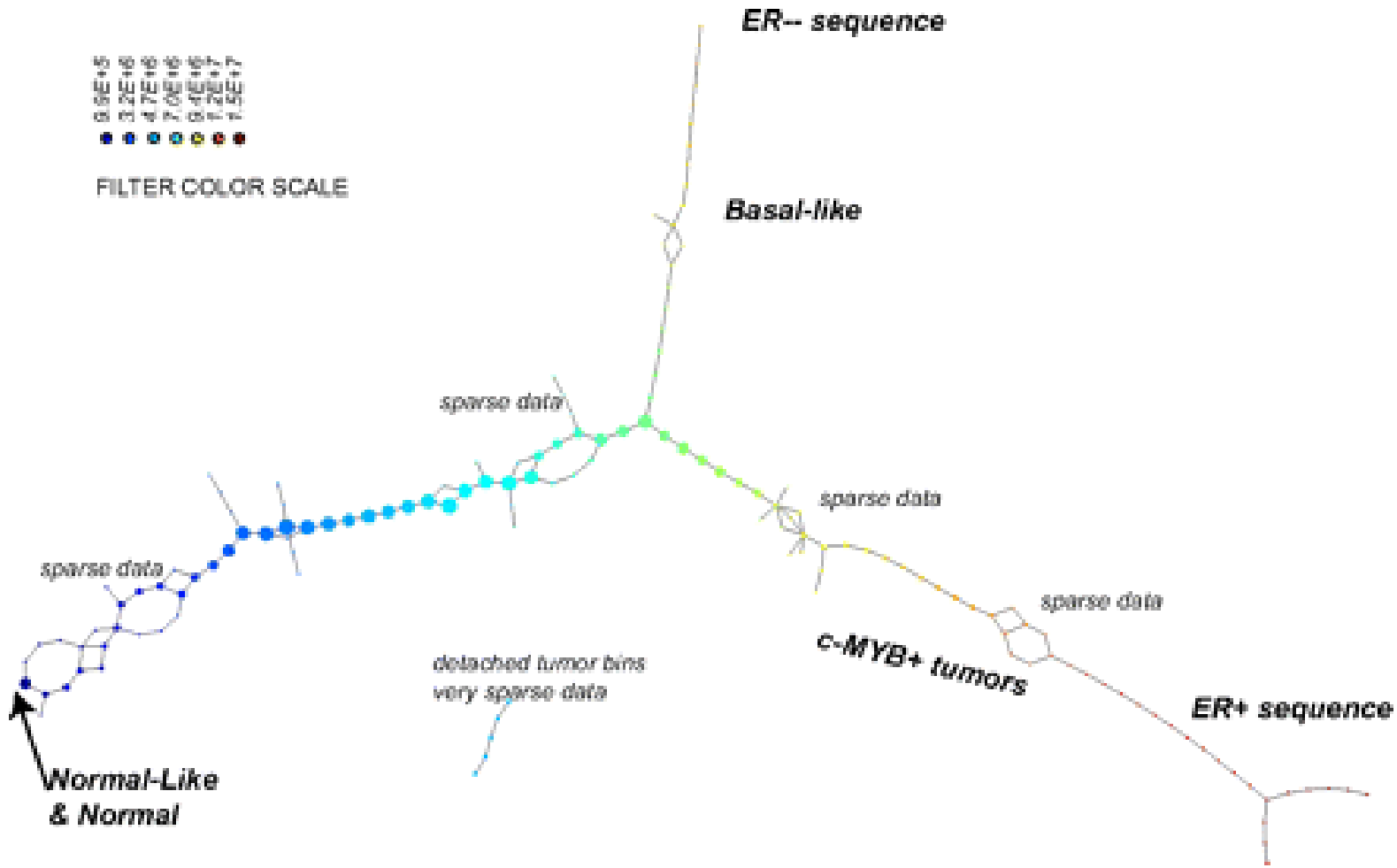
Choose $f : X \to [0, \infty), x \mapsto d(v_1, x)$

and $\mathcal{U}$ an open cover of $Z = [0, \infty)$ with no triple overlaps

Mapper output for starfish sample



Mapper output for breast cancer microarray gene expression data



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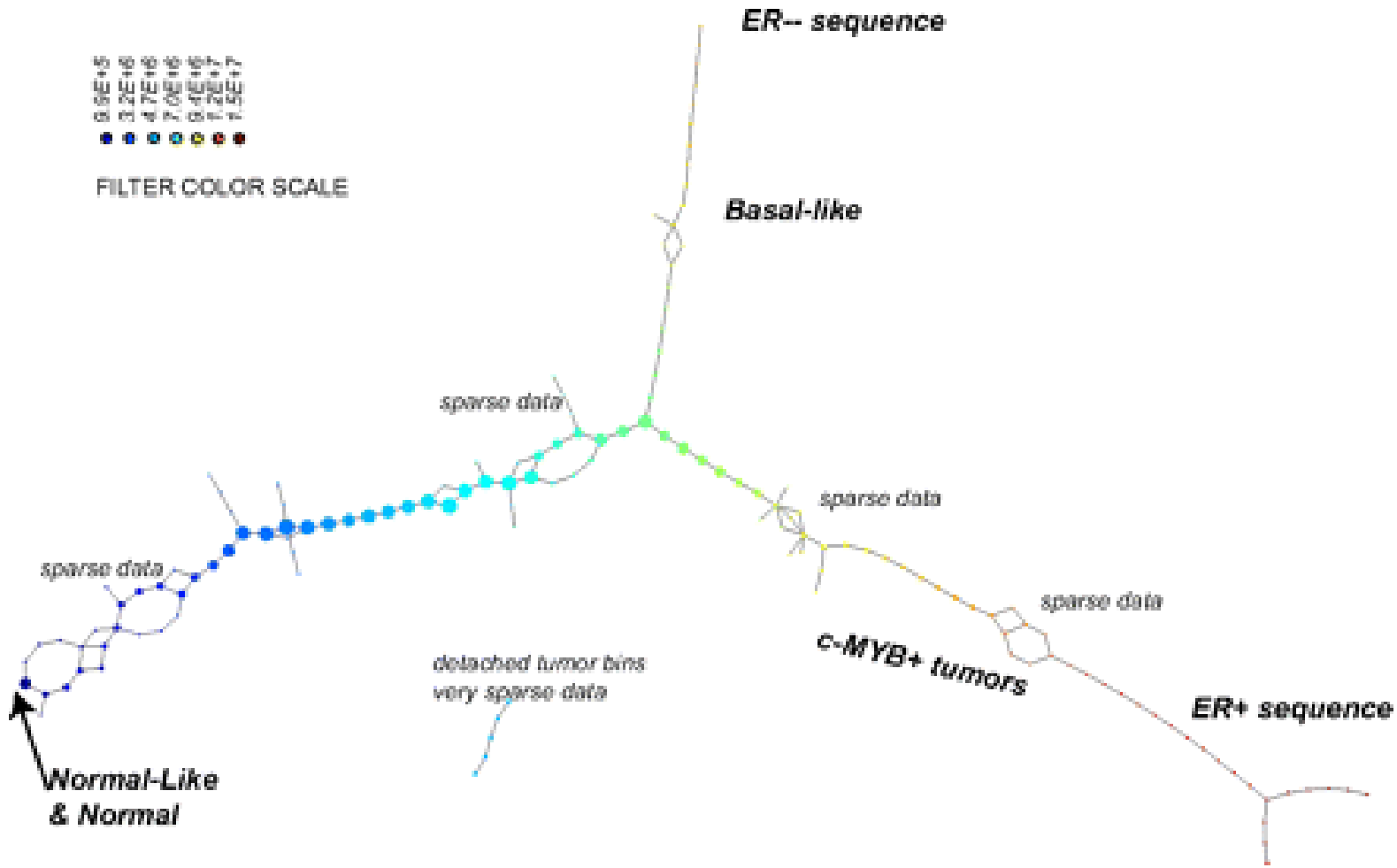
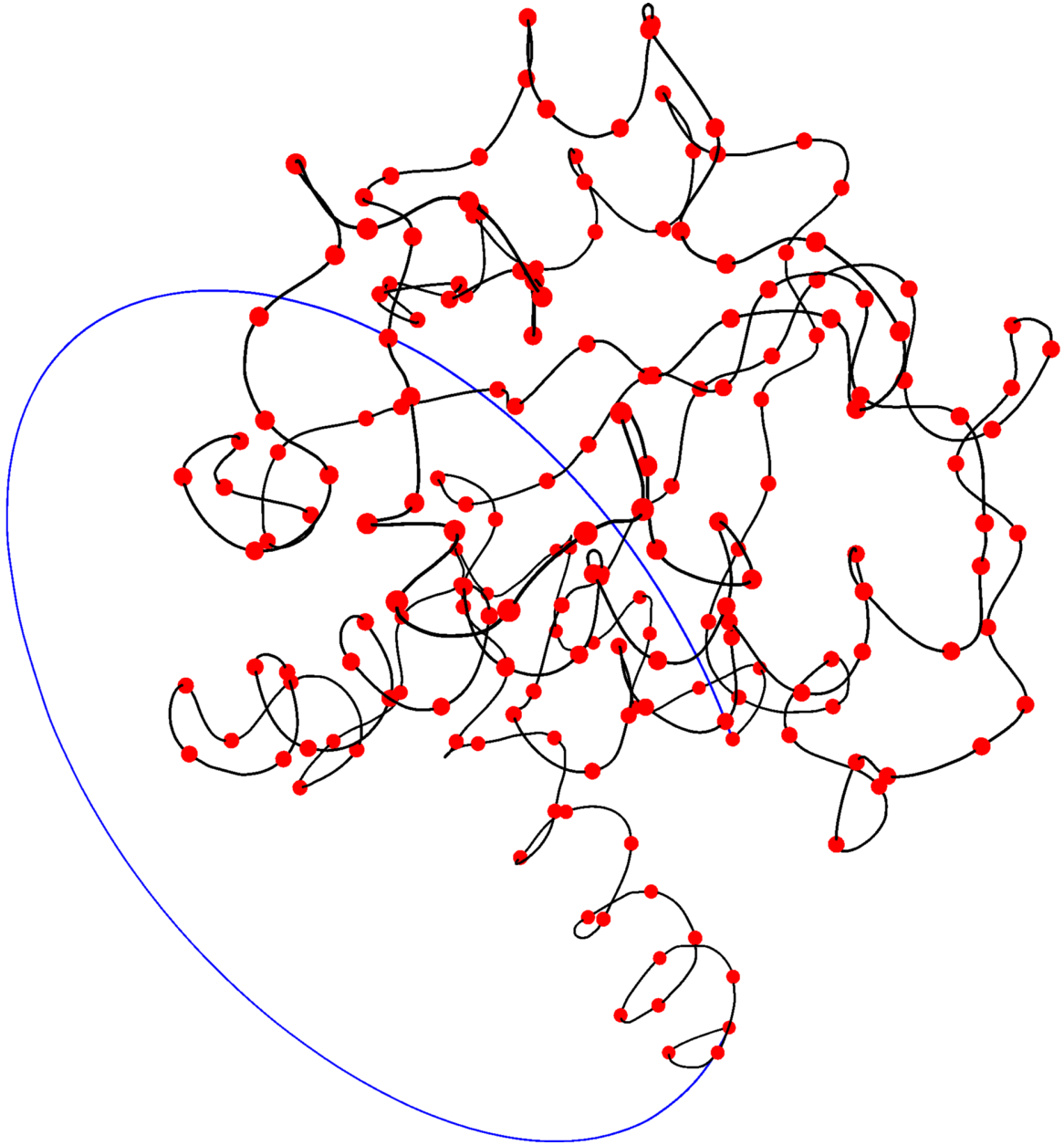


Nicolau, Levine, Carlsson (PNAS, 2011): identified a subgroup of ER+ breast cancers. These patients exhibit 100% survival.

1V2X protein backbone: Is it knotted?



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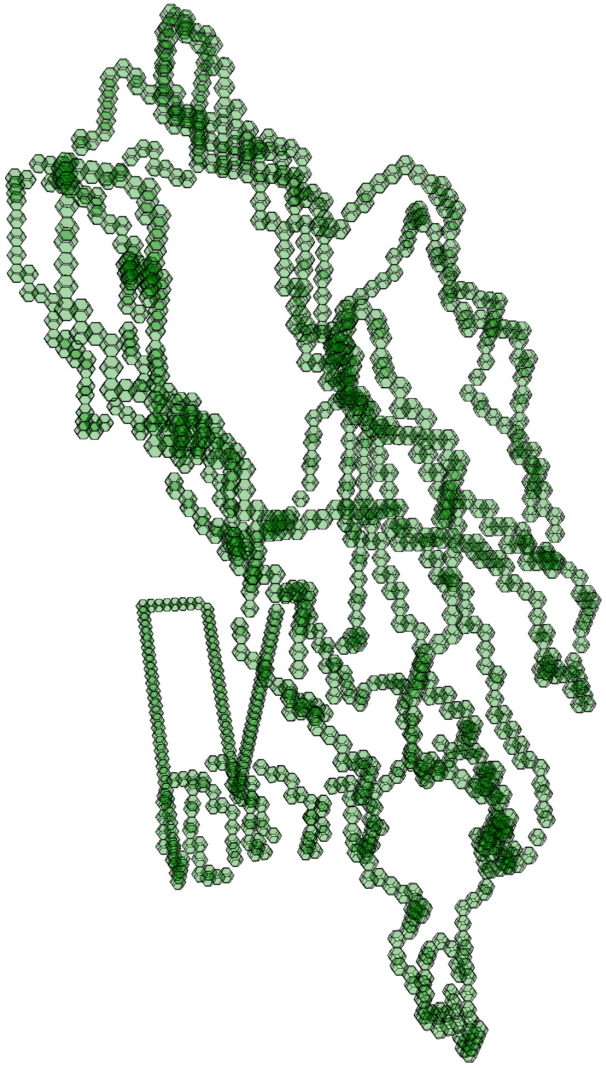

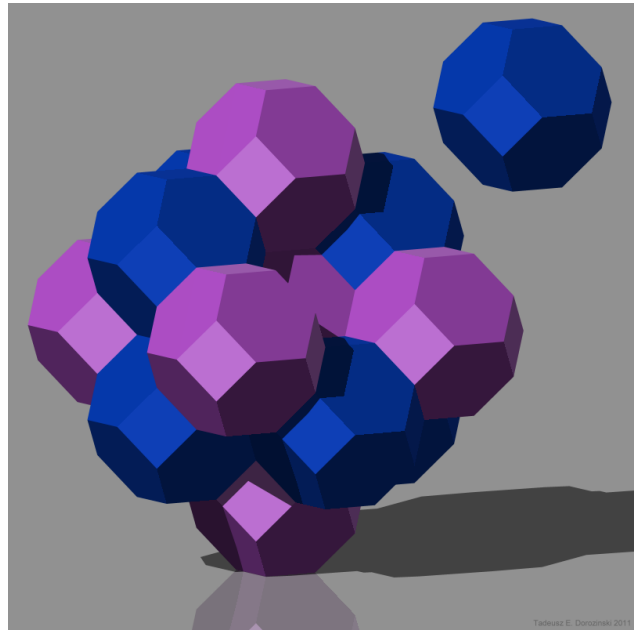

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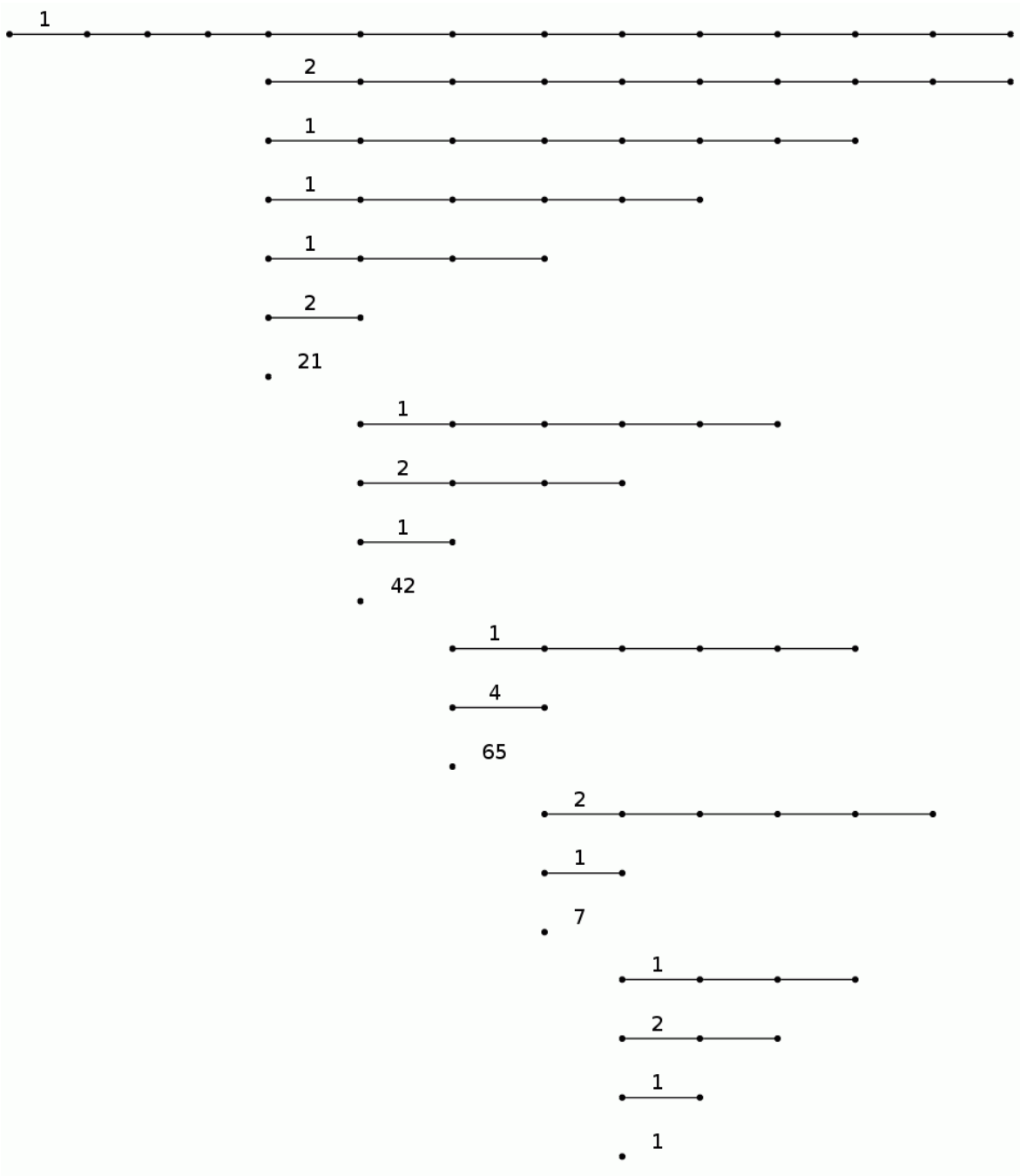
Persistent β_1



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$$\pi_1 Y \cong \langle\, x, y \mid yx^{-1}yxy^{-1}x \,\rangle$$

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Pure permutahedral complex of dimension 3
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gap> FundamentalGroup(Y);
[ f1^-3*f2*f1^2*f2*f1, f1 ]
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min value of n to distinguish between knots on c crossings

c	3	4	5	6	7	8	9	10	11	12	13	14
n	2	2	3	3	3	3	5	5	6	6	7	7

Brendel, E., Juda, Mrozek