Third test: Monday 27 March

Recall A game involves

- n players
- a set S_i of strategies for player i
- a payoff function

$$\nu_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$

for player i, $1 \le i \le n$.

Example $n = 2$

$S_1 = \{H, T\}$, $S_2 = \{H, T\}$

$\nu_1(H,H) = 1$ $\nu_2(H,H) = -1$	$\nu_1(H,T) = -1$ $\nu_2(H,T) = 1$
$\nu_1(T,H) = -1$ $\nu_2(T,H) = 1$	$\nu_1(T,T) = 1$ $\nu_2(T,T) = -1$

Defn A mixed strategy is a choice of probabilities

$p_{i,s}$ = probability that player i plays strategy $s \in S_i$.

for $1 \leq i \leq n$, $s \in S_i$ satisfying

$$p_{i,s} \geq 0, \quad \sum_{s \in S_i} p_{i,s} = 1.$$

Notation Suppose $S_i = \{s_1, s_2, \ldots, s_k\}$ and set

$$\underline{p}_i = (p_{i,s_1}, p_{i,s_2}, \ldots, p_{i,s_k}).$$

Define the expected payoff for player i to be the function

$$\Sigma_i(\underline{P}_1, \underline{P}_2, \cdots, \underline{P}_n) = E(v_i)$$

$$= \sum_{\substack{x_1 \in S_1 \\ x_2 \in S_2 \\ \vdots \\ x_n \in S_n}} p_{1,x_1} p_{2,x_2} \cdots p_{n,x_n} v_i(x_1, x_2, \cdots, x_n).$$

A mixed Nash equilibrium occurs if, having played the game, no player unilaterally benefits by changing her/his mixed strategy (the other mixed strategies being fixed).

Theorem (J. Nash) In any game with finitely many players and finite pure strategy sets S_i, there exists at least one mixed Nash equilibrium.

Example For our 2-player game above, $S_1 = \{H, T\}$, $S_2 = \{H, T\}$

$$E_i(\underline{P}_1, \underline{P}_2) =$$

$$P_{1H} P_{2H} v_i(H,H) + P_{1T} P_{2H} v_i(T,H) + P_{1H} P_{2T} v_i(H,T) + P_{1T} P_{2T} v_i(T,T)$$

$$E_1(P_{1H}, P_{1T}, P_{2H}, P_{2T}) =$$

$$P_{1H} P_{2H} - P_{1T} P_{2H} - P_{1H} P_{2T} + P_{1T} P_{2T}$$

Example In our 2-player game an example of a mixed Nash equilibrium is the strategy:

$p_{1H} = \frac{1}{2}$ $p_{1T} = \frac{1}{2}$ $p_{2H} = \frac{1}{2}$ $p_{2T} = \frac{1}{2}$

Outline proof of Nash's Theorem

Consider

$$C = \{(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_n)\} \subseteq \mathbb{R}^{|S_1| + \dots + |S_n|}$$

where $\underline{p}_i \in \mathbb{R}^{|S_i|}$ is the probability distribution for player i, $1 \leq i \leq n$.

Now C is closed, bounded and convex, since $p_{i,s} \geq 0$,

$$\sum_{s \in S_i} p_{i,s} = 1 \quad .$$

Thus any continuous function $f : C \to C$ has at least one fixed point (Brouwer's theorem).

For any given $(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_n) \in C$ define $\underline{q}_i \in \mathbb{R}^{|S_i|}$ to be the probability distribution that maximizes

$$\max_{\underline{q}} \Sigma_i(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_{i-1}, \underline{q}, \underline{p}_{i+1}, \dots, \underline{p}_n) \qquad (*)$$

Now define $f : C \to C$ by

$$f(\underline{p}_1, \underline{p}_2, \dots, \underline{p}_n) = (\underline{q}_1, \underline{q}_2, \dots, \underline{q}_n)$$

This f has a fixed point.

But such a fixed point is a mixed Nash equilibrium.

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Slight problem: The quantity $\underline{q}_i$ that maximizes (*) may not be unique. Thus f is not a well-defined function.

To solve this problem replace (*) by

$$\max_{\underline{q}} \left(\Sigma_i (\underline{P}_1, \ldots, \underline{P}_{i-1}, \underline{q}, \underline{P}_{i+1}, \ldots, P_n) - \| \underline{P}_i - \underline{q} \|^2 \right) \qquad (**)$$