Third Topology Test:
Monday 27 March

Last time:

$\pi_1(S^1) = \mathbb{Z}$

Fundamental Theorem of Algebra

Any polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$ and of degree $n > 0$ has at least one zero in $\mathbb{C}$.

Proof: Since $a_n \neq 0$ a scalar multiple of the polynomial has the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

Let's suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

For $\lambda \geq 0$ define a map

$$f_\lambda : S^1 \longrightarrow S^1$$

by

$$f_\lambda(z) = \frac{p(\lambda z)}{|p(\lambda z)|}$$

Any two maps f_λ, $f_{\lambda'}$ are homotopic via the homotopy

$$H_t(z) = \frac{p(((1-t)\lambda + t\lambda')z)}{|p(((1-t)\lambda + t\lambda')z)|}$$

Note that for $\lambda = 0$ we have f_0 is a constant function, and

thus has winding number 0.

Exercise: For large λ we have that $f_\lambda(z)$ is homotopic to $g_n : S^1 \to S^1$, $z \mapsto z^n$.

But g_n has winding number n.

we $g_n \simeq f_\lambda \simeq f_0$, and winding number of f_0 is 0.

But homotopic maps have the same winding number.

$n \neq 0$ is a contradiction.

$\blacksquare$

Game Theory

A game involves

- n players
- a set S_i of strategies for player i.
- a payoff function

$$v_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$

for player i, $i = 1, 2, \ldots, n$.

Example 2 players, Mary and John. They want to go to either the cinema (c) or soccer match (s) together.

$v_1(c,c) = 2$ $v_2(c,c) = 1$	$v_1(c,s) = 0$ $v_2(c,s) = 0$	$S_1 = \{c, s\}$ $S_2 = \{c, s\}$
$v_1(s,c) = 0$ $v_2(s,c) = 0$	$v_1(s,s) = 1$ $v_2(s,s) = 2$	

Example 2 2 players, each places a coin on the table. Player 1 wants coins to be the same. player 2 wants coins to be different.

$$S_1 = \{H, T\}$$

$$S_2 = \{H, T\}$$

Payoff :

$v_1(H,H) = 1$ $v_2(H,H) = -1$	$v_1(H,T) = -1$ $v_2(H,T) = 1$
$v_1(T,H) = -1$ $v_2(T,H) = 1$	$v_1(T,T) = 1$ $v_2(T,T) = -1$

In a pure strategy game each player decides, beforehand, on a strategy to play.

A pure Nash equilibrium occurs if, having played the game, no player benefits from unilaterally changing his/her choice of strategy.

Example 1 There are two pure Nash equilibria: both go to the cinema, or both go to soccer.

Example 2 There is no pure Nash equilibrium in this game.