A second problem

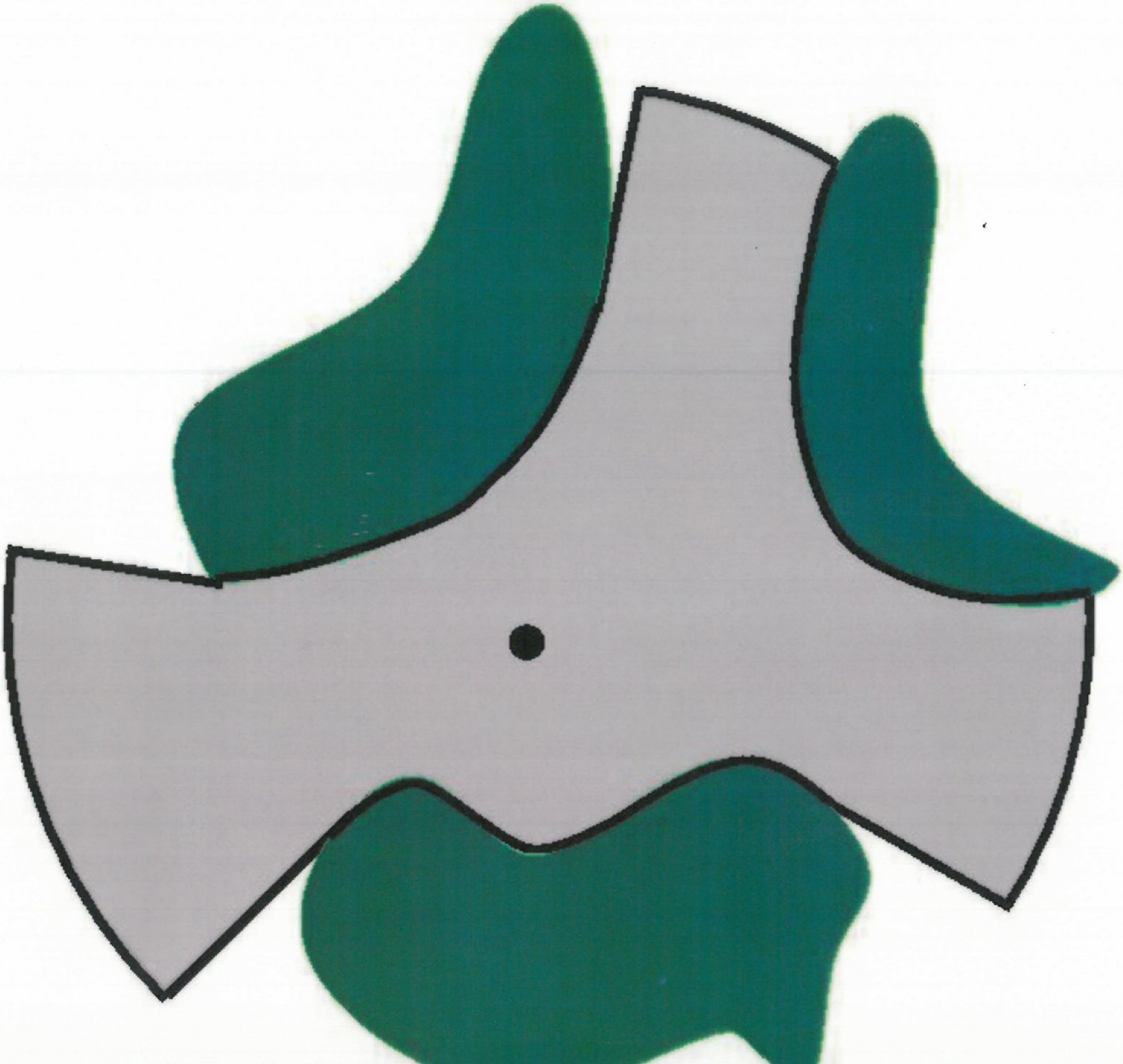
Taken from research paper by Yuliy Barishnikov, and Robert Christ.

A Texas farmer invests in thousands of cheap sensors, and spreads them out over the ranch.

When activated from the farmhouse a sensor counts the number of cows within a distance r say, and within line of sight.

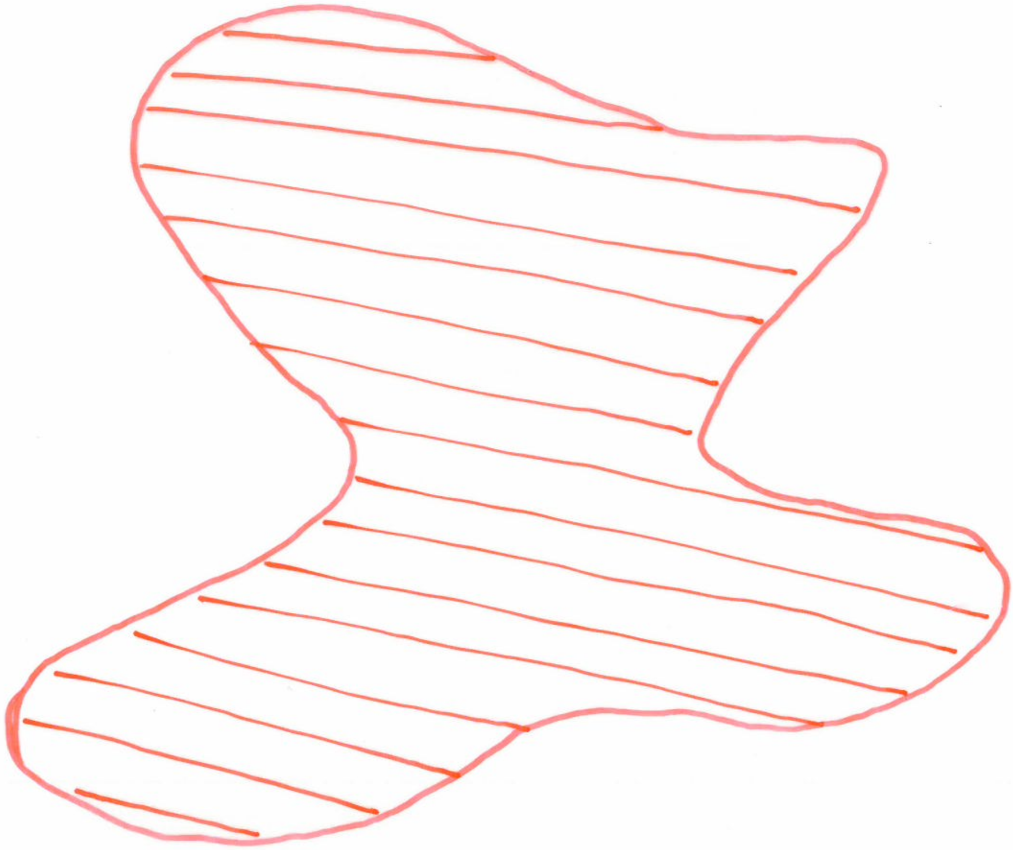
The number, and the sensor id is returned to the farmhouse.

How can the farmer determine the the number of cows on the ranch from this data?



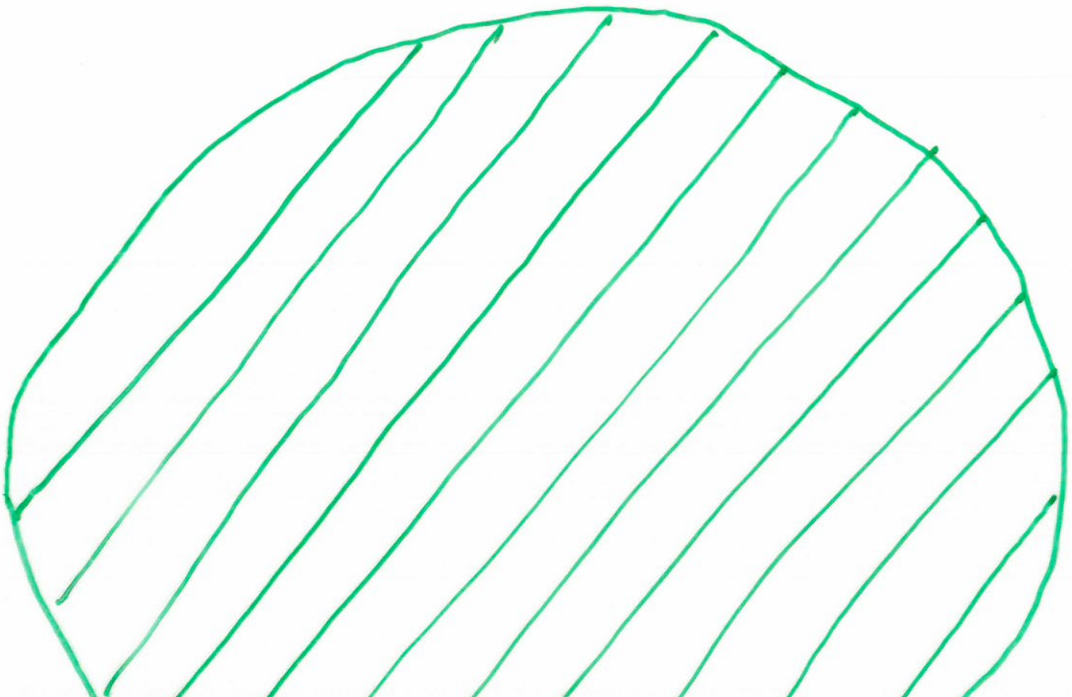
A point cow is detected by all sensors in the gray region; it is partially hidden by some green objects.

U_1

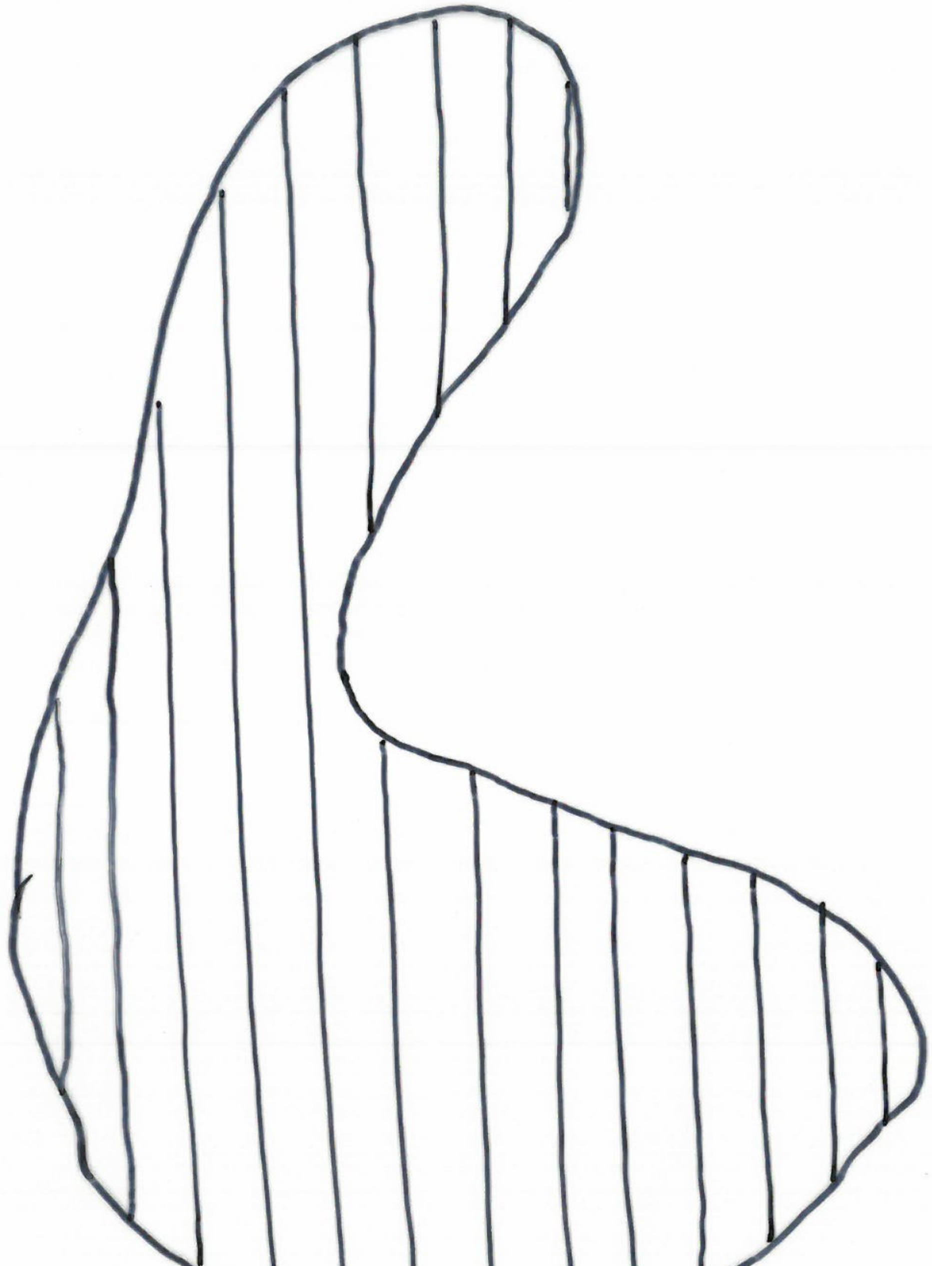


U_1 = visibility region of cow 1

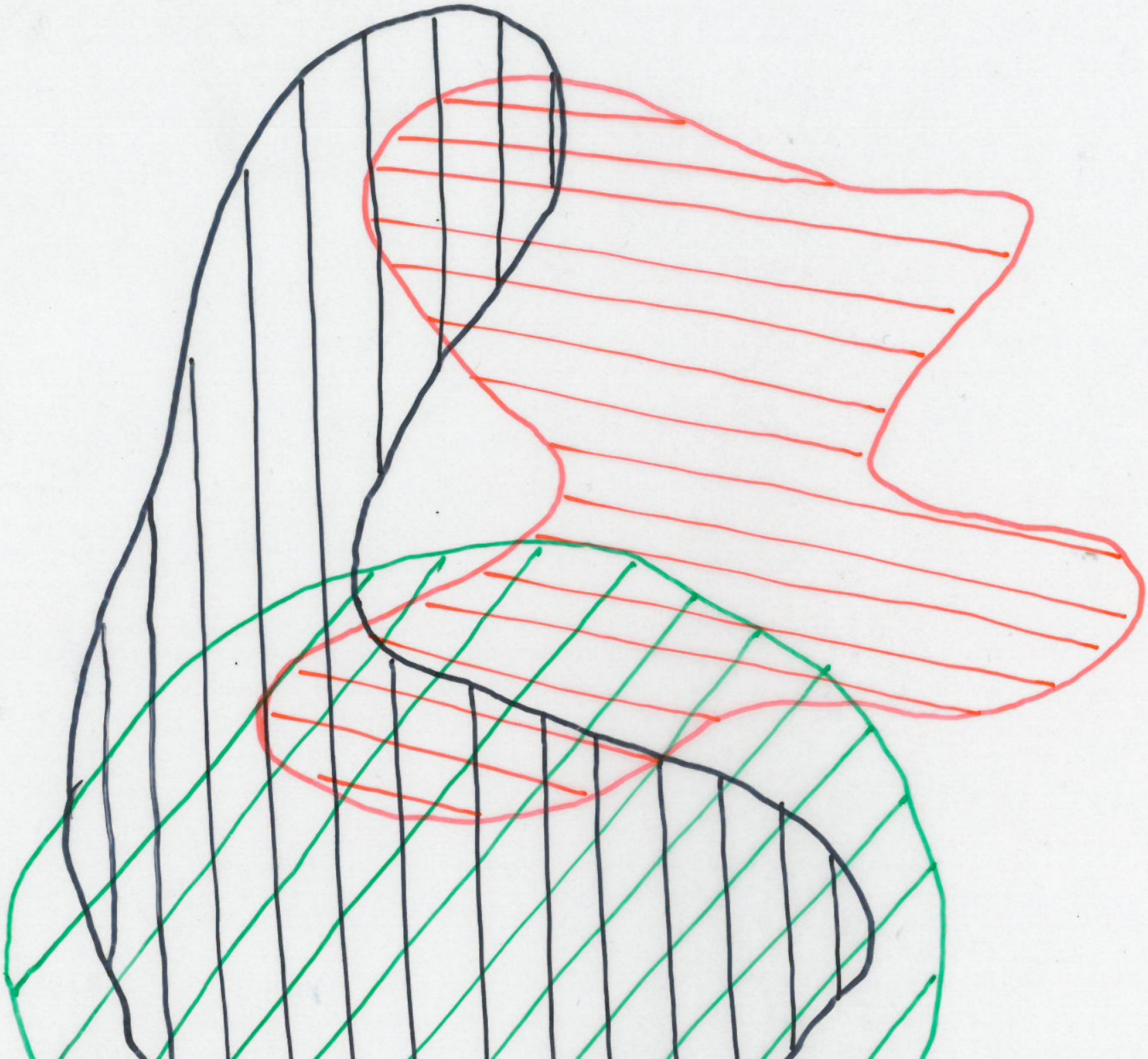
U_2



u_3

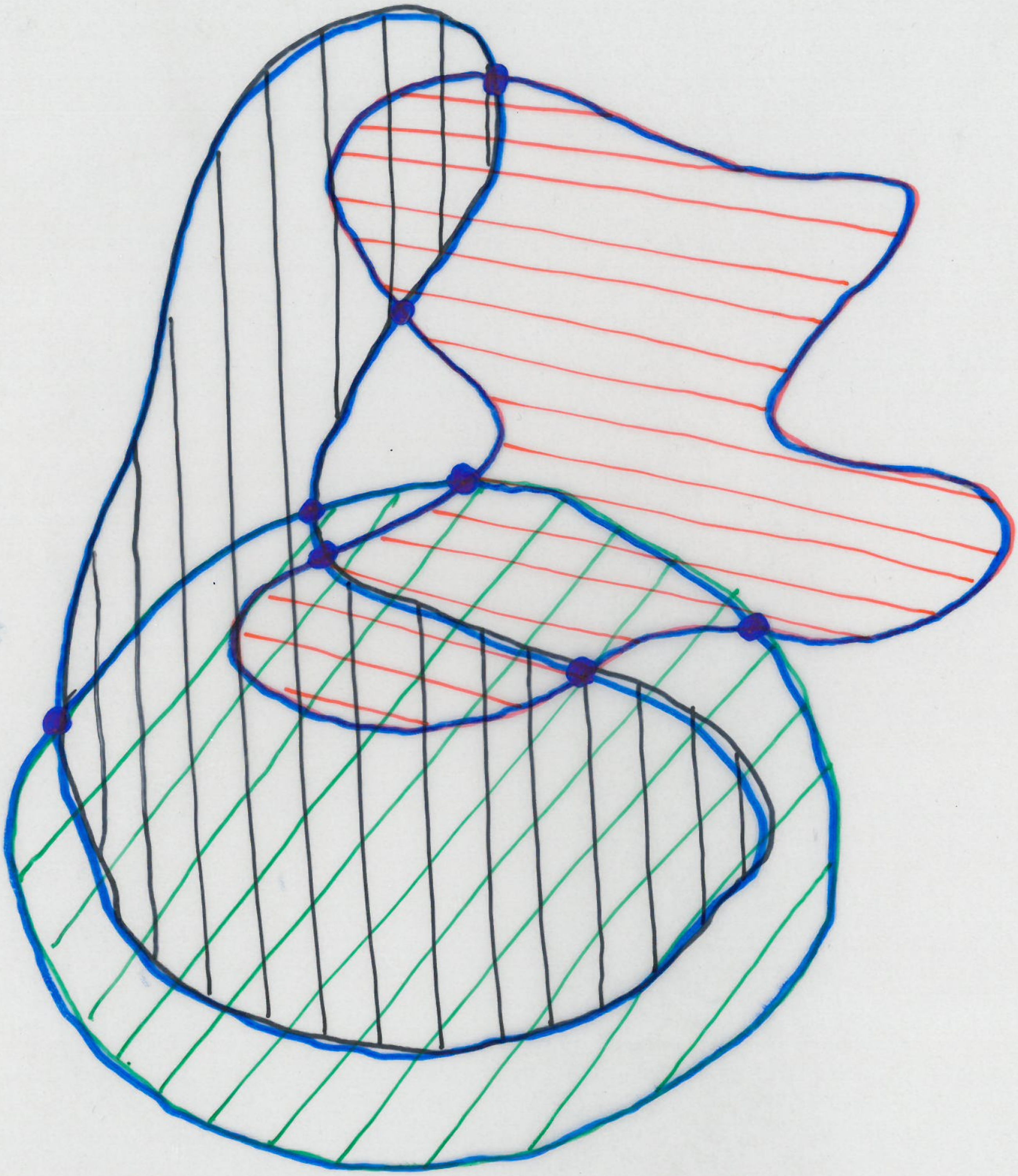


U_1 U_2 U_3



U_1 = visibility region of cow 1

$X = U_1 \cup U_2 \cup U_3$ = entire ranch



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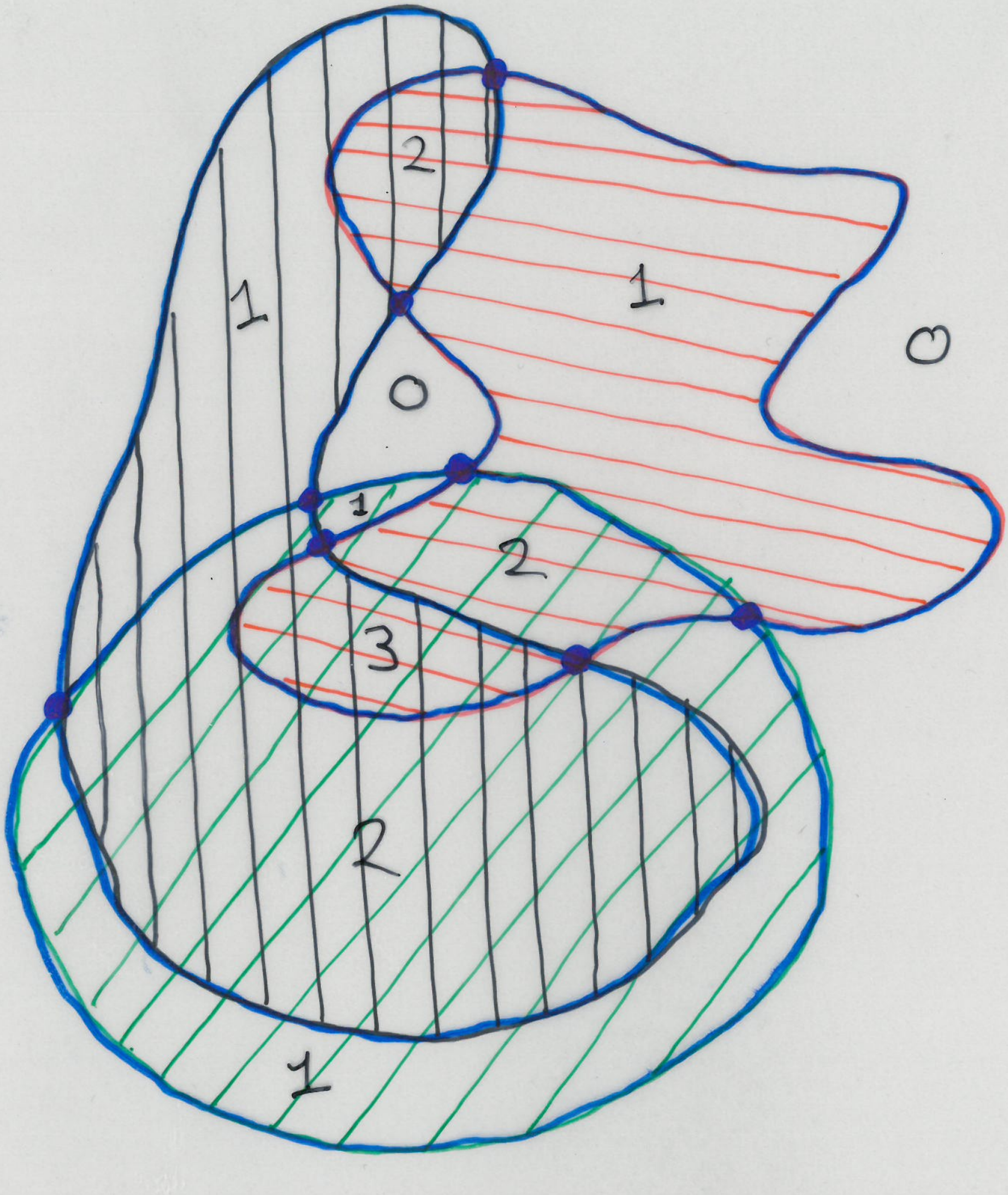

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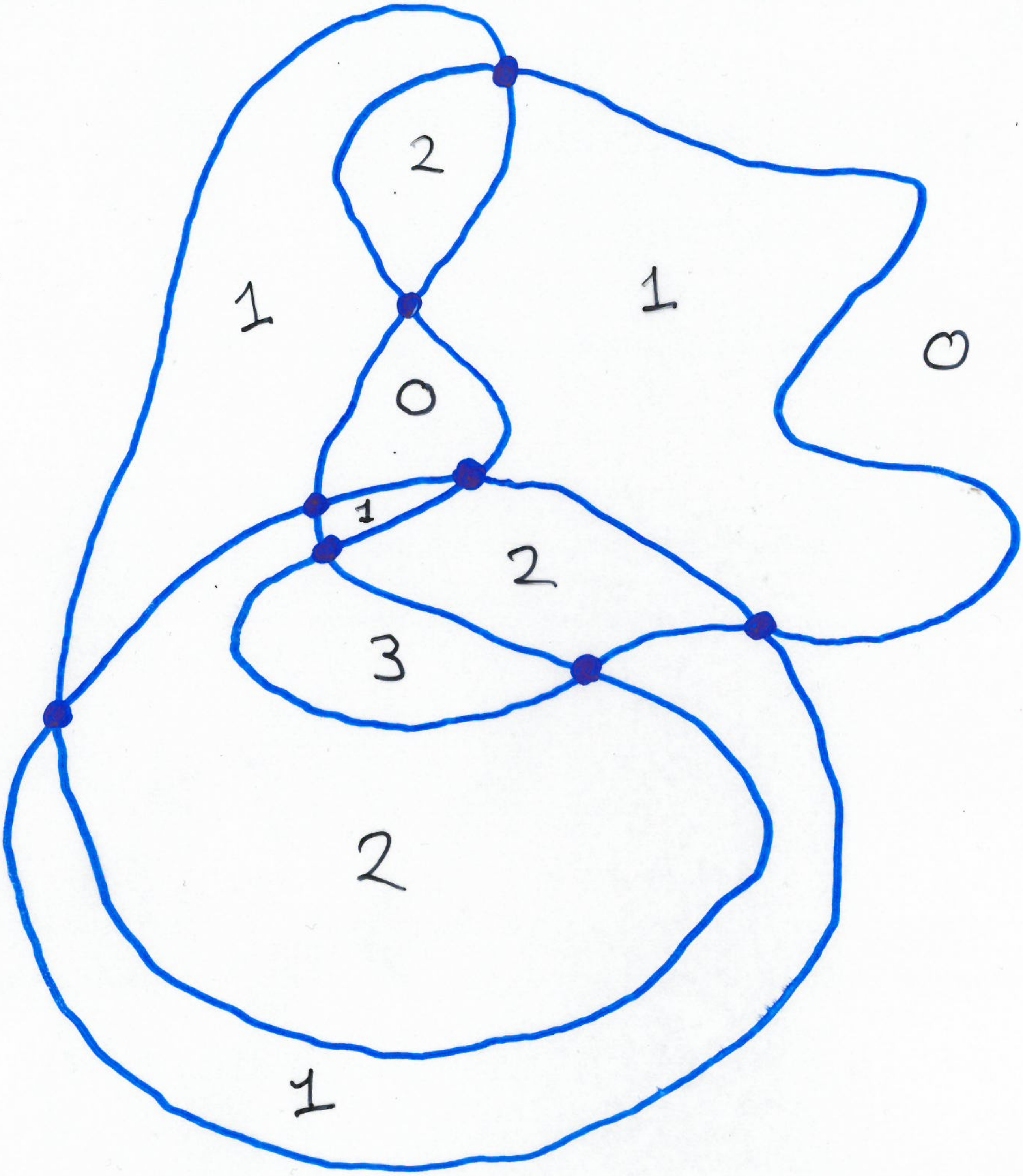


Given such a union

$$X = U_1 \cup U_2 \cup \cdots \cup U_t$$

we say that a function

$$w : X \longrightarrow \mathbb{Z}$$

is a <u>weight function</u> if it is constant on any given vertex, on any given edge, and on any given face.

<u>Example</u> The function

$$w(x) = \left| \{ i : x \in U_i \} \right|$$

(= number of colours uses to colour x)

Definition Given a weight function

$$\omega : X = U_1 \cup \cdots \cup U_t \longrightarrow \mathbb{Z}$$

we define the Euler integral

$$\int_X \omega \, d\chi = \sum_v \omega(v) - \sum_e \omega(e) + \sum_f \omega(f)$$

where v, e, f range over vertices, faces edges, and where $\omega(f)$ means $\omega(x)$ with $x \in f$.

Example (Texas ranch)

$$\omega = |\{ i : i \in U_i \}|$$

Then

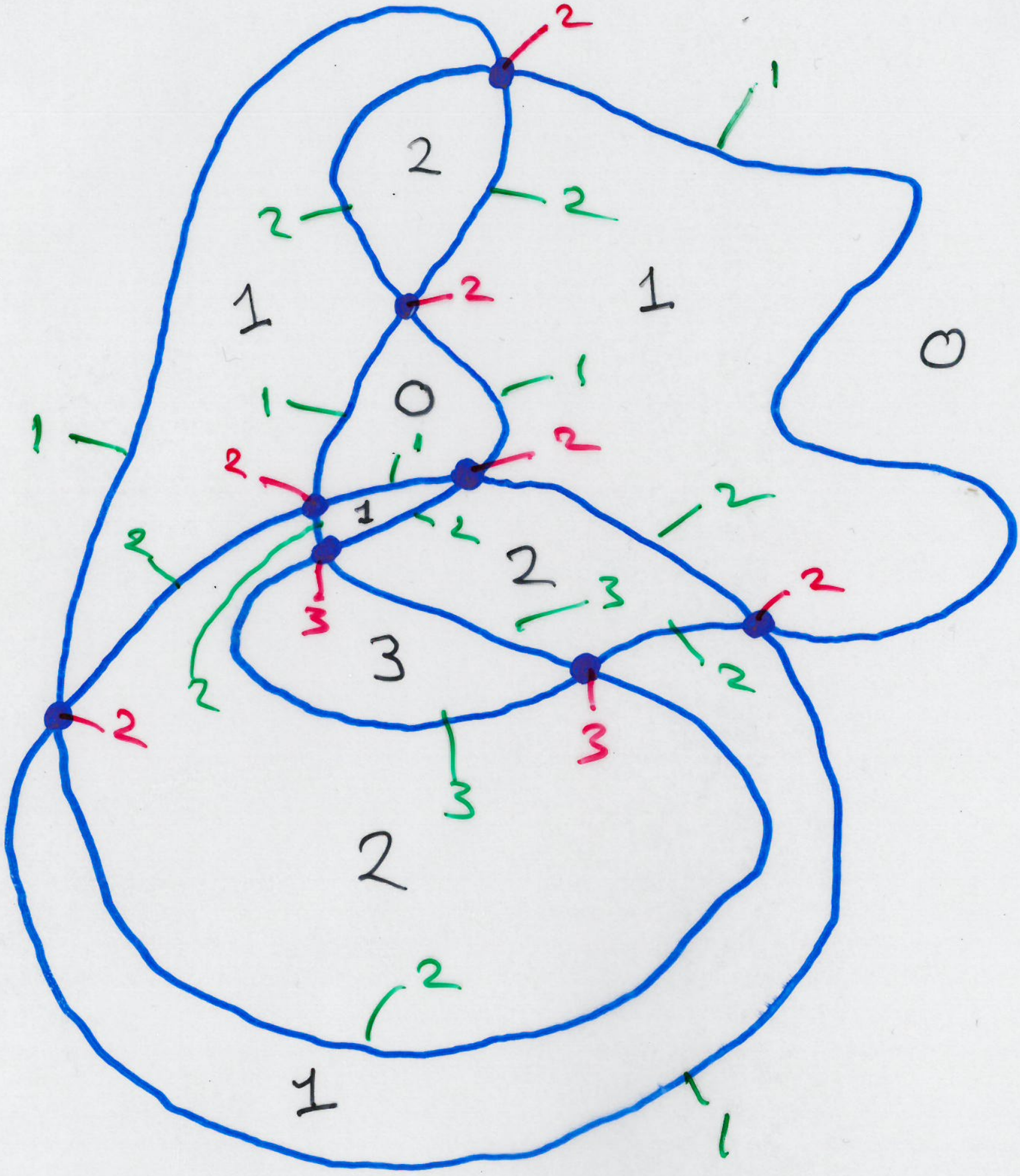
$$\int_X \omega \, d\chi = (6 \times 2 + 2 \times 3)$$

$$- (6 \times 1 + 8 \times 2 + 2 \times 3)$$

$$+ (4 \times 1 + 3 \times 2 + 3)$$

$$= 18 - 28 + 13 = 3$$

$X = U_1 \cup U_2 \cup U_3$ = entire ranch



<u>Theorem</u> Let $X \subseteq \mathbb{R}^2$ be a region with subregions $U_1, U_2, \dots, U_t \subseteq \mathbb{R}^2$ such that

$$X = U_1 \cup U_2 \cup \cdots \cup U_t .$$

Let $\omega(x) = |\{ i : i \in U_i \}|$

suppose that each U_i has the same Euler characteristic

$\chi(U_i) = c$. Then

$$t = \frac{1}{c} \int_X \omega \, d\chi .$$

<u>Proof</u> Let

$$1_{U_i} : X \to \mathbb{Z}$$

be defined as

$$1_{U_i} = \begin{cases} 1 & \text{if } x \text{ is in } U_i \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_X w \, dx = \int_X \left(\sum_{1 \le i \le t} 1_{U_i} \right) dx$$

$$= \sum_{1 \le i \le t} \left(\int_X 1_{U_i} \, dx \right)$$

Think

$$= \sum_{1 \le i \le t} \chi(U_i)$$

$$= tc.$$

QED