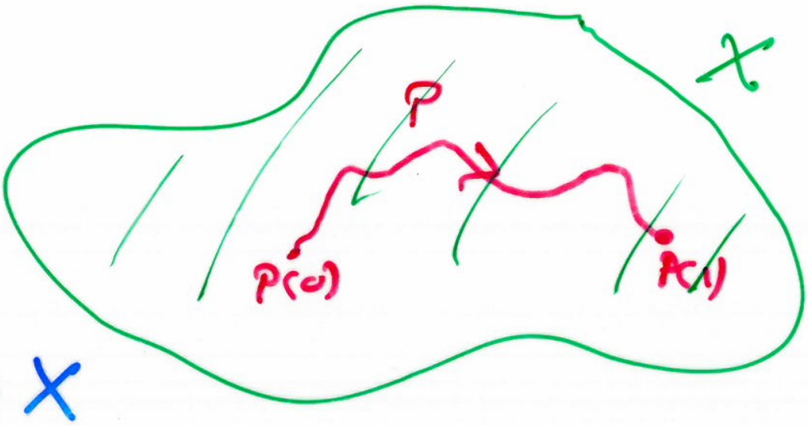
Third Topology Test
27 March (Monday)

Topology and group theory are intimately related.

Let X be a topological space.

A continuous function $p : [0,1] \to X$ is called a path in X.



Choose $x_0 \in X$

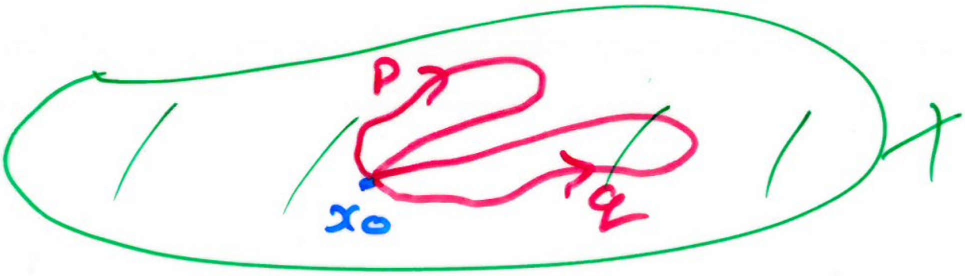
A path $p : [0,1] \to X$ with $p(0) = p(1) = x_0$ is called a loop at x_0.

Given two loops $p, q : [0,1] \to X$ at x_0 we can combine them to form a new loop at x_0

$$p * q : [0,1] \to X$$

by defining

$$p*q(s) = \begin{cases} p(2s), & 0 \le s \le \frac{1}{2} \\ q(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$



This "multiplication" $p*q$ does not satisfy the axioms of a group. It does not even have an identity element.

Two loops $p, q : [0,1] \to X$ at x_0 are homotopic rel x_0 if there is a continuous map

$$H : [0,1] \times [0,1] \longrightarrow X, \; (s,t) \mapsto H_t(s)$$

with

$$H_0(s) = p(s)$$
$$H_1(s) = q(s)$$
$$H_t(0) = H_t(1) = x_0 \text{ for all } t \in [0,1].$$

So $H_t(s)$ is a loop at x_0 for all $t \in [0,1]$.

Homotopy rel x_0 is an equivalence relation on loops at x_0. Let $[p]$ denote the equivalence class of p.

Let

$$\pi_1(X, x_0) = \left\{ [p] : \begin{array}{l} p : [0,1] \to X \text{ is} \\ \text{a loop at } x_0 \end{array} \right\}.$$

Theorem (Henri Poincaré)

$\pi_1(X, x_0)$ is a group with multiplication

$$[p] * [q] = [p * q].$$

Proof Not difficult. See Armstrong's book.

Terminology

We call $\pi_1(X, x_0)$ the fundamental group of X.

Example $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

$1 \in S^1$.

$\pi_1(S^1, 1) \cong \mathbb{Z}$

See Armstrong for full details.

The idea :

Consider the loop

$$p_1 : [0,1] \longrightarrow S^1, \quad \theta \longmapsto e^{2\pi i \theta}$$

Then

$$p_1 * p_1 : [0,1] \to S^1, \quad \theta \longmapsto e^{4\pi i \theta}$$

And

$(p_1 * p_1) * p_1$ is homotopic rel 1

to

$$p_3 : [0,1] \longrightarrow S^1, \quad \theta \longmapsto e^{6\pi i \theta}.$$

In general we have a

loop

$$p_n : [0,1] \longrightarrow S^1, \quad \theta \mapsto e^{2n\pi i \theta}$$

for $n \in \mathbb{Z}$.

One needs to show;

1) $[P_n] \neq [P_m]$ if $n \neq m$

2) Any loop $q : [0,1] \to S^1$ at 1 is homotopic rel 1 to some P_n. We say that q has winding number n.

3) $[P_n * P_m] = [P_{n+m}]$.