$D^n = \{ x \in \mathbb{E}^n : \|x\| \le 1 \}$

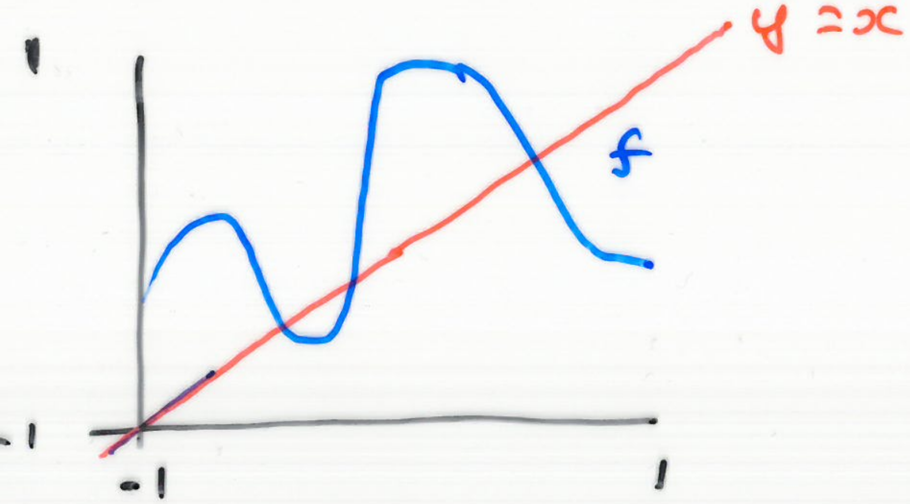
Brouwer's Theorem

Any continuous map $f: D^n \to D^n$ has at least one fixed point, $f(x) = x \in D^n$.

$n=1$

$D^1 = [-1, 1]$

We picture a map $f: D^1 \to D^1$ by its graph:



A fixed point is a point where the blue graph intersects the red line.

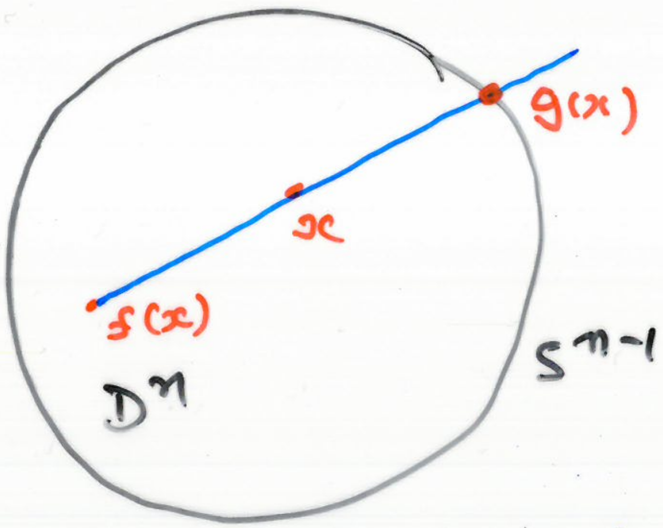
Proof of Brouwer's Theorem ($n \geq 1$)

Let $f: D^n \to D^n$ be continuous.

Suppose f has no fixed point.

Then we can define a continuous map

$$g: D^n \longrightarrow S^{n-1}, \quad x \longmapsto g(x)$$



where $g(x)$ is the point in S^{n-1} where the ray from $f(x)$ through x intersects S^{n-1}.

Note that g is continuous.

Let $h : S^{n-1} \longrightarrow D^n$, $x \longmapsto x$

Now $gh : S^{n-1} \to S^{n-1}$ is the identity on S^{n-1}. Thus

$$gh \simeq 1_{S^{n-1}}$$

Now $hg : D^n \longrightarrow D^n$ is homotopic to the identity on D^n, via the homotopy

$$H(x,t) = x + t(g(x) - x).$$

$H(x,0) = x$

$H(x,1) = g(x) = hg(x)$

So $hg \simeq 1_{D^n}$.

Therefore D^n is homotopy equivalent to S^{n-1}.

Thus, our major theorem implies

$$\chi(D^n) = \chi(S^{n-1}).$$

But $\chi(D^n) = 1$

and $\chi(S^{n-1}) = 0$ or 2.

This contradiction proves the theorem. $\boxed{\text{u}}$

<u>Theorem</u> (Frobenius - Perron)

Let A be a real $n \times n$ matrix with entries $a_{ij} > 0$ for all i, j. Then A has a positive eigenvalue. Moreover, there is a corresponding eigenvector $v = (x_1, \dots, x_n)$ with each $x_i \geq 0$.

Proof of theorem

Define $\sigma : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i$

$$\Delta^{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \begin{array}{l} x_i \geq 0 \\ \text{and } \sum_{i=1}^{n} x_i = 1 \end{array} \right\}$$

Define

$$g : \Delta^{n-1} \to \Delta^{n-1}, \quad x \to \frac{1}{\sigma(Ax)} Ax$$

now g is continuous, and Δ^{n-1} is (homeomorphic to) D^{n-1}.

Brouwer's Theorem says that g has a fixed point $x \in \Delta^{n-1}$

$$x = g(x) = \frac{1}{\sigma(Ax)} Ax$$

So $Ax = \sigma(Ax) . x$

Therefore x is an eigenvector of A with Eigenvalue $\delta(Ax)$.

$\boxed{4}$