Two maps

$$f, g : X \to Y$$

are homotopic if there is a map $H : X \times [0,1] \to Y$, $(x,t) \mapsto H_t(x)$ such that

$$H_0(x) = f(x)$$

and $H_1(x) = g(x)$.

Defn Two spaces X, Y are homotopy equivalent if there exist maps

$$f : X \to Y, \quad g : Y \to X$$

with

$$fg \simeq 1_Y \quad \text{and} \quad gf \simeq 1_X$$

where $\simeq$ means homotopic, and $1_X : X \to X$ is the identity on X, and $1_Y : Y \to Y$ is the identity on Y.

Example 1 If X and Y are homeomorphic then they are also homotopy equivalent. If $f: X \to Y$ is a homeomorphism then there is a map $g: Y \to X$ with

$$fg = 1_Y \quad \text{and} \quad gf = 1_X$$

Example 2 $X = \mathbb{C} \setminus \{0\}$ and $Y = S^1$ are homotopy equivalent.

We have

$$g: Y = S^1 \longrightarrow X, \quad z \longmapsto z$$

$$f: X \longrightarrow Y = S^1, \quad z \longmapsto \frac{1}{|z|} z$$

Well $fg = 1_Y$, and so $fg \simeq 1_Y$.

To see that $gf \simeq 1_X$ we use the homotopy

$$H : X \times [0,1] \to X, \quad (z,t) \mapsto \left(\frac{1-t}{|z|} + t\right) z$$

and note that

$$H_1(z) = 1_X(z) = z$$

$$H_0(z) = gf(z).$$

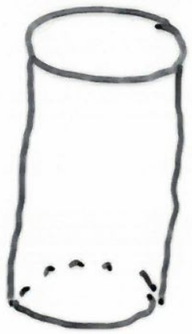
<u>Major</u> <u>Theorem</u>

Let X, Y be spaces with triangulations. If X and Y are homotopy equivalent then

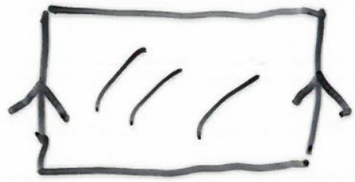
$$\chi(X) = \chi(Y).$$

Illustration The circle S^1 is homotopy equivalent to the cylinder.

$S^1 \times [0,1] =$

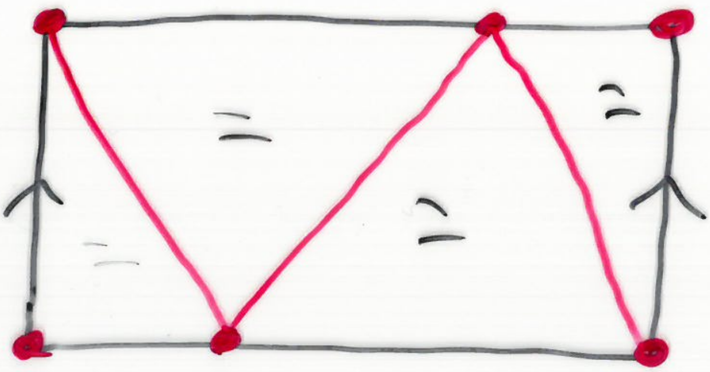


$=$



S^1

$\chi(S^1) = 3 - 3 = 0$



$\chi(\text{cylinder}) = 4 - 8 + 4 = 0.$

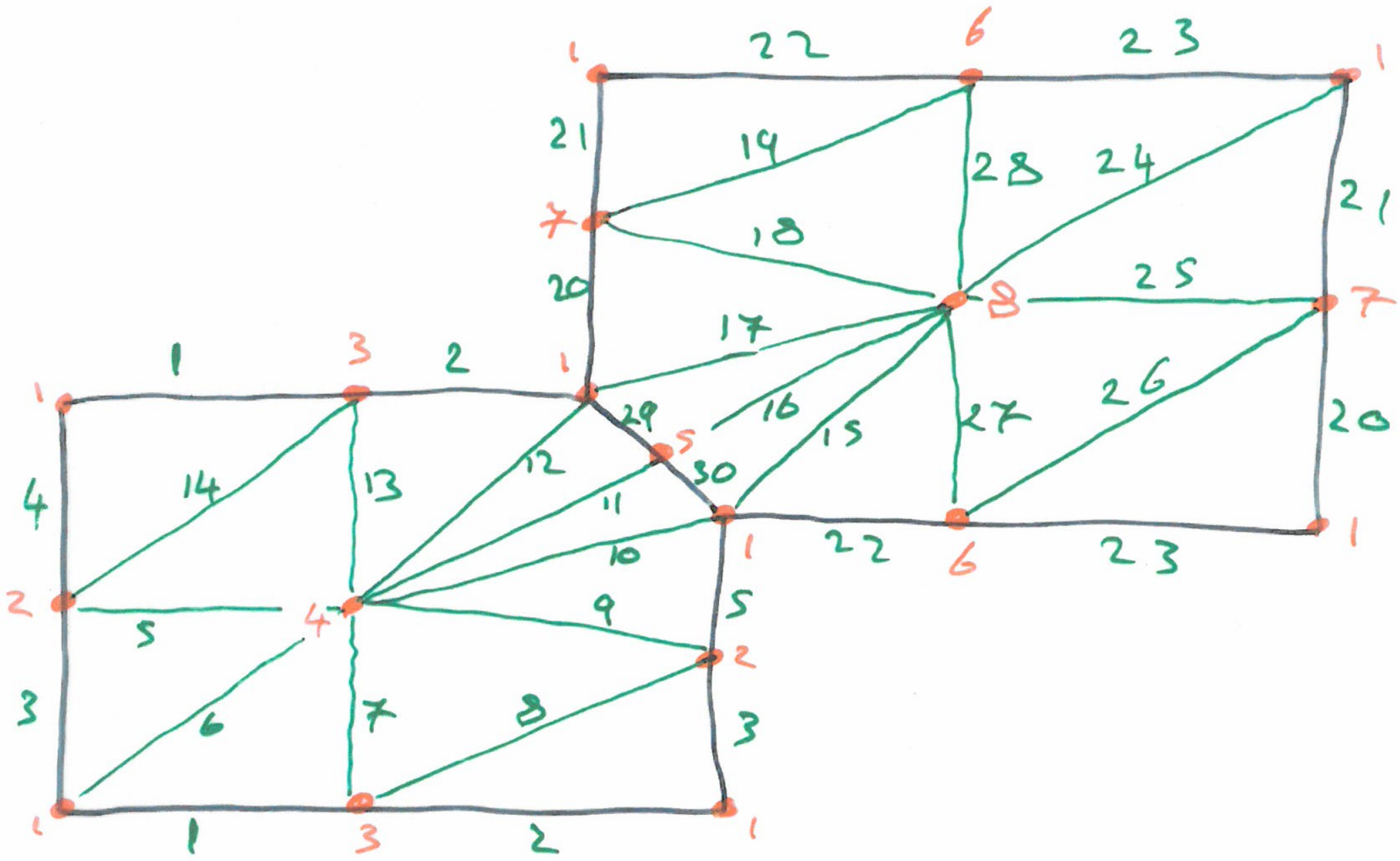
Illustration The n-simplex Δ^n is homotopy equivalent to a singleton space $\{*\}$.

So $\chi(\Delta^n) = \chi(\{*\}) = 1$.

Illustration

$$\chi(S^n) = \chi(\Delta^{n+1}) \pm 1 = 1 \pm 1$$

$$= \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$



$\chi(\text{Double Torus}) = 8 - 30 + 20 = -2$

Recall

$$D^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$$

Brouwer's Theorem For any continuous map $f : D^n \to D^n$ there exists at least one $x \in D^n$ such $f(x) = x$.

Defn For any map $f : X \to X$ a point $x \in X$ satisfying $f(x) = x$ is called a fixed point.