"Homotopy" is a key notion used in the proof of the topological invariance of Euler characteristic.

Defn Two maps $f: X \to Y$ and $g: X \to Y$ are homotopic if there exists a continuous map

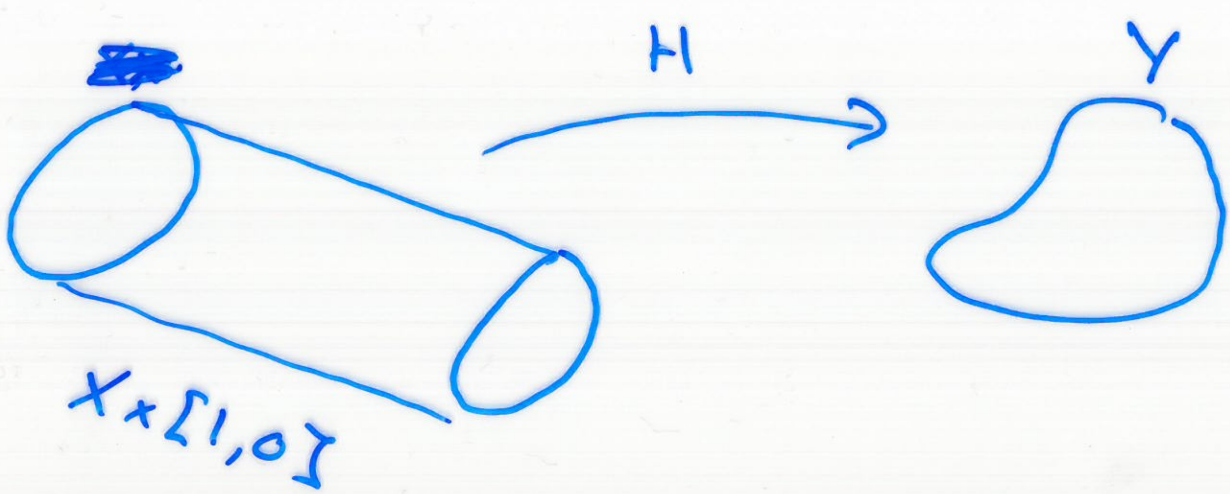
$$H : X \times [0,1] \longrightarrow Y, \quad (x,t) \longmapsto H_t(x)$$

such that

$H_0(x) = f(x)$, and

$H_1(x) = g(x)$

for all $x \in X$.



We can think of $H_t(x)$ as a family of maps

$$H_t(x) : X \to Y, \quad x \mapsto H_t(x)$$

We refer to H as the <u>homotopy</u>, and we write

$$f \simeq g.$$

To understand homotopy we need to understand the topology on $X \times [0,1]$.

Let $\mathcal{B}$ be the collection of all sets

$$U \times J = \{(u, j) : u \in U, j \in J\}$$

with U any open subset of X and J any open subset of $[0,1]$. A set is <u>open</u>

in $X \times [0,1]$ if it is a union of sets in $\mathcal{B}$. [$\mathcal{B}$ is called a <u>basis</u> for the topology.]

Intuition

$$H : X \times [0,1] \longrightarrow Y$$

is continuous if a small change in x and a small change in t produces only a small change in $H(x,t) = H_t(x)$.

Example Let $Y \subseteq \mathbb{E}^2$ be a convex set. Let X be any topological space.

Any two maps $f : X \to Y$, $g : X \to Y$ are homotopic.

To see this we define a homotopy

$$H : X \times [0,1] \to Y,$$

$$(x,t) \longmapsto f(x) + t\,(g(x) - f(x))$$
$$= (1-t) f(x) + t\, g(x).$$

describes the line joining $f(x)$ and $g(x)$.

So $H(x,t) \in Y$ since Y is convex.

Also $H(x,0) = f(x)$
$H(x,1) = g(x)$.

So $f \simeq g$ since H is continuous!

Proposition For fixed spaces X, Y homotopy is an equivalence relation on the collection of all maps from X to Y.

Proof For any $f: X \to Y$ we have $f \simeq f$ (reflexive) thanks to the homotopy

$$H_t(x) = f(x).$$

For any $f: X \to Y$, $g: X \to Y$ if $f \simeq g$ then there is a homotopy $H_t(x)$ with $H_0(x) = f(x)$ and $H_1(x) = g(x)$. To see that $g \simeq f$ define

$$H'_t(x) = H_{1-t}(x)$$

(symmetric)

For transitivity Let $f, g, h : X \to Y$ be such that

$f \simeq g$ and $g \simeq h$.

So we have homotopies

$H_t(x)$, $H_0(x) = f(x)$, $H_1(x) = g(x)$

$H'_t(x)$, $H'_0(x) = g(x)$, $H'_1(x) = h(x)$.

To see that $f \simeq h$ define

$$H''_t(x) = \begin{cases} H_{2t}(x), & 0 \leq t \leq \frac{1}{2} \\ H'_{2t-1}(x), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$\square$

Let $[f]$ denote the homotopy class of a map $f : X \to Y$.
Let $[X, Y]$ denote the collection of homotopy classes of maps $X \to Y$.