Linear Algebra

There is no linear isomorphism

$\phi : \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$

Topology

There is no homeomorphism

$\phi : \mathbb{R} \to \mathbb{R}^2.$

(To see this, note $\mathbb{R} \setminus \{0\}$ has two connected components, whereas $\mathbb{R}^2 \setminus \{\phi(0)\}$ has just one connected component.)

Linear Algebra

There is no linear surjection

$\phi : \mathbb{R} \twoheadrightarrow \mathbb{R}^2.$

($\dim(\operatorname{im}\phi) + \dim(\ker\phi) = \dim \mathbb{R}$
so $\dim(\operatorname{im}\phi) \leq 1$.
But $\dim(\mathbb{R}^2) = 2$.)

Topology

There exists a surjective continuous function

$$\phi : \mathbb{R} \twoheadrightarrow \mathbb{R}^2 .$$

i.e. $\mathrm{im}(\phi) = \mathbb{R}^2$. ϕ is called a space-filling curve.

How can we construct such a surjective map ϕ?

Let Δ be an equilateral triangular region of $\mathbb{R}^2 = \mathbb{E}^2$ of side 1

$\Delta =$  $\subseteq \mathbb{E}^2$

Theorem (Peano) There exists a surjective continuous function

$$f : [0,1] \twoheadrightarrow \Delta$$

<u>Proof</u> We'll first construct a sequence of continuous functions

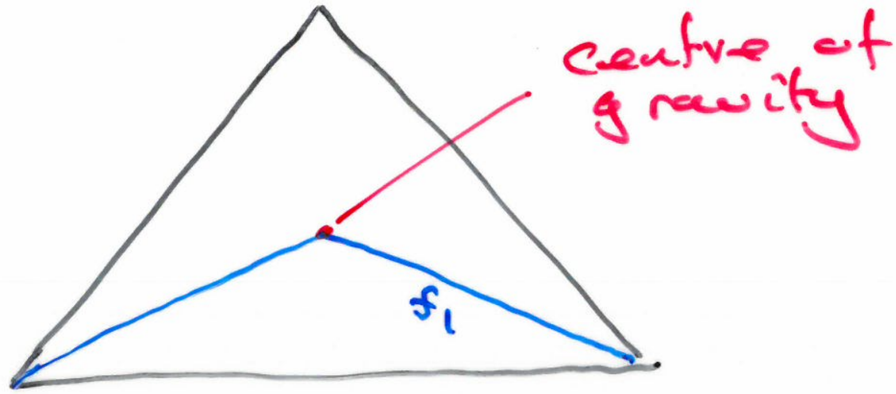
$$f_1 : [0,1] \longrightarrow \Delta$$

$$f_2 : [0,1] \longrightarrow \Delta$$

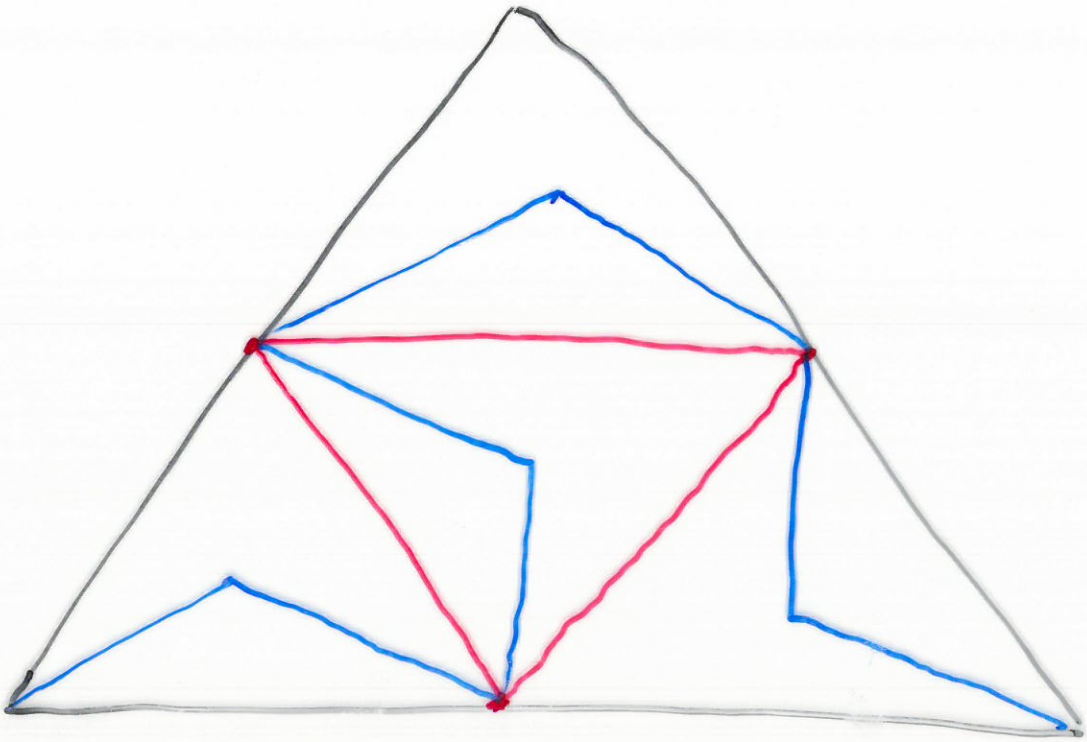
$$f_3 : [0,1] \longrightarrow \Delta$$

$$\vdots$$

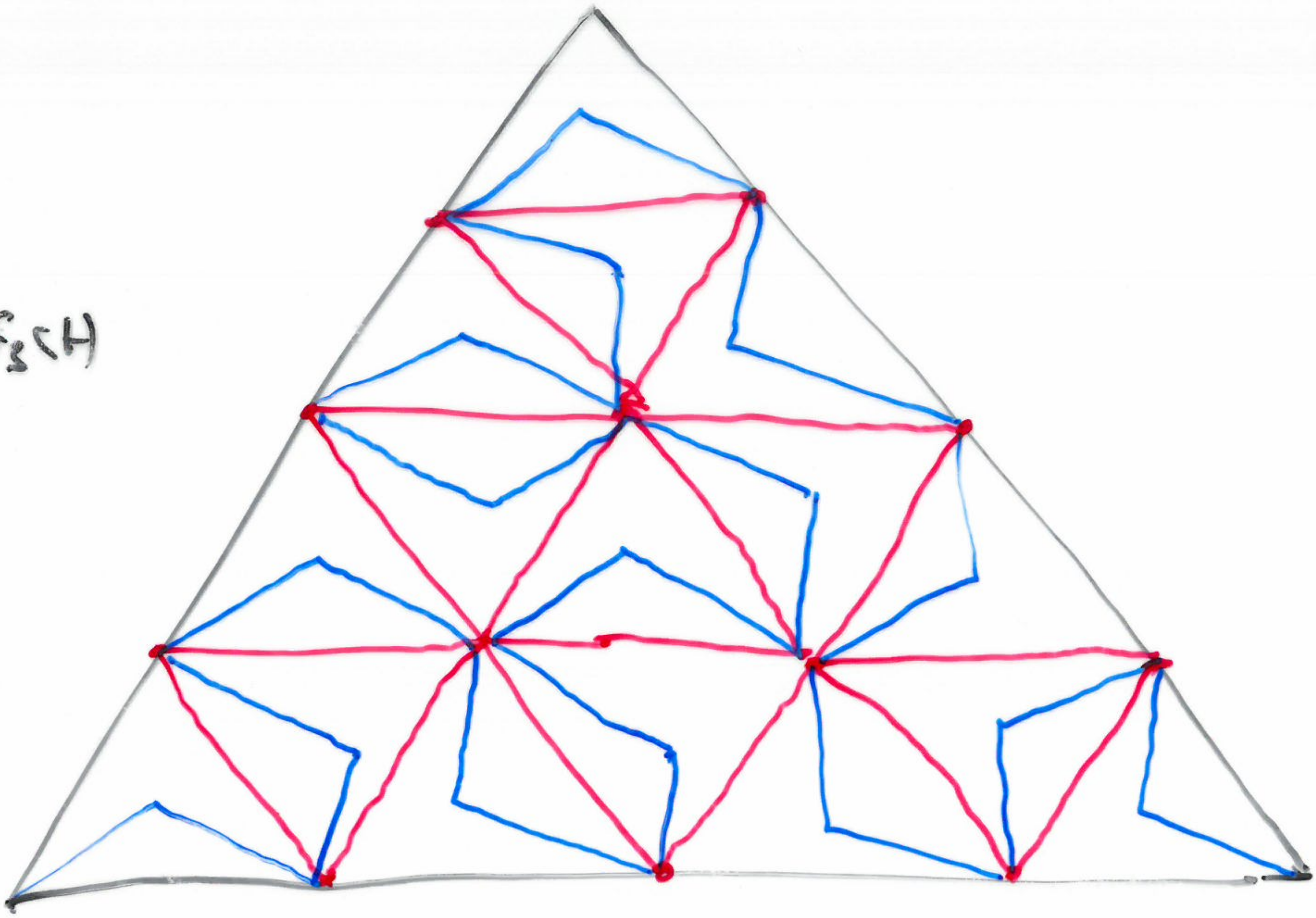
$f_1(t)$



$f_2(t)$



$f_3(t)$



For $f_n : [0,1] \to \Delta$ we subdivide Δ into 4^{n-1} red triangles, and the image of f_n inside each red triangle looks just like f_1.

To complete the proof of the theorem we:

1) Define f to be the limit of $f_1, f_2, f_3, \ldots$

2) Need to show that the limit is continuous and onto. (Compactness is used here.)

Aside: Some basics on limits

Consider

$a_1 = 1, a_2 = 1\frac{1}{2}, a_3 = 1\frac{5}{6}, \ldots, a_n = a_{n-1} + \frac{1}{n}, \ldots$

So $\lim\limits_{n\to\infty} a_n - a_{n-1} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\lim\limits_{n\to\infty} a_n =$ does not exist

a_n = area of first n boxes

$$> \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

<u>Defn</u> A sequence of points $a_1, a_2, \ldots$ in $\mathbb{E}^k$ is said to be a Cauchy sequence if, for any $\varepsilon > 0$ there is an N such that

$$\|a_m - a_n\| < \varepsilon$$

for all $m, n \geq N$.

Theorem Any Cauchy sequence $a_1, a_2, \ldots$ in $\mathbb{E}^k$ has a limit $\lim_{n\to\infty} a_n$.

Return to Peano

we've constructed $f_1 : [0,1] \to \Delta$, $f_2 : [0,1] \to \Delta$, ...

Fix $t \in [0,1]$

For any $m > n$ we have that $f_m(t)$ and $f_n(t)$ both lie in a single equilateral triangle of side $\frac{1}{2^{n-1}}$.

So

$f_1(t), f_2(t), f_3(t), \ldots$

is a Cauchy sequence.

The sequence thus has a limit which we denote by $f(t)$.

Next time: need to show that $f : [0,1] \to \Delta$ is continuous and onto.