MA342 Topology

See

http://hamilton.nuigalway.ie

for textbook, homework sheet, assessment and other details

Lecturer: Graham Ellis

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Basic Topology by M.A. Armstrong

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with a view to

- understanding the above description of topology,
- understanding classical and modern motivations for topology.

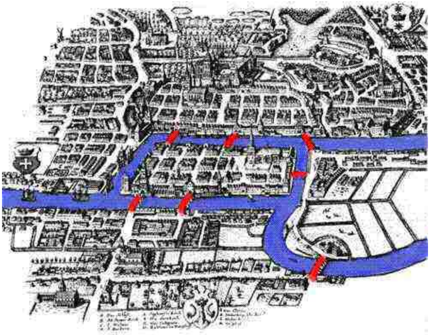
Leonard Euler

In 1736 he published a paper:

Solutio problematis ad geometriam situs pertinentis

(The solution of a problem relating to the geometry of position)

Königsberg around Euler's time



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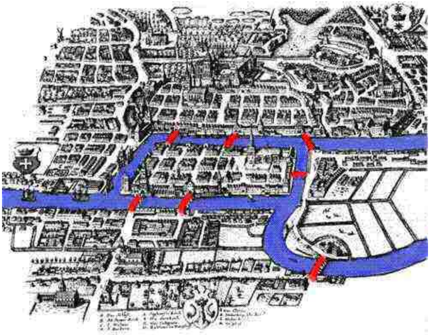
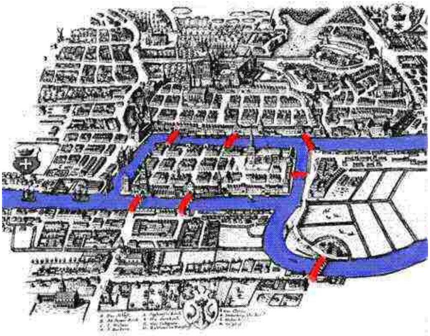


The problem:

Decide whether it is possible to follow a path that crosses each bridge exactly once and returns to the starting point.

The problem involved a different type of geometry where distance was not relevant.



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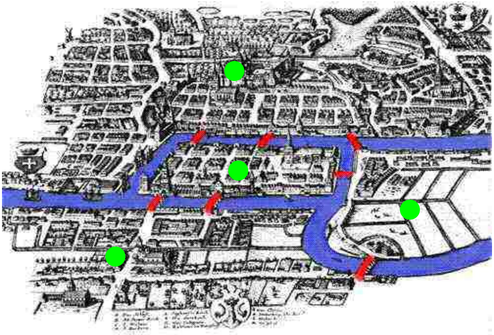

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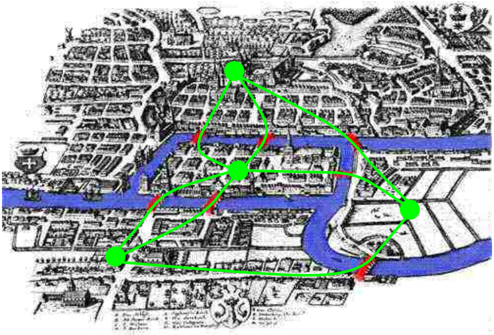

To analyze the problem Euler placed a vertex in each land mass.

And inserted an edge between vertices for each bridge connecting the corresponding land masses.

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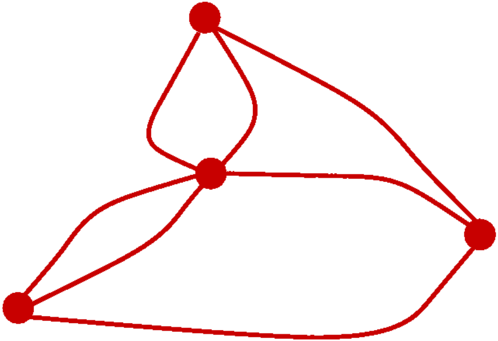

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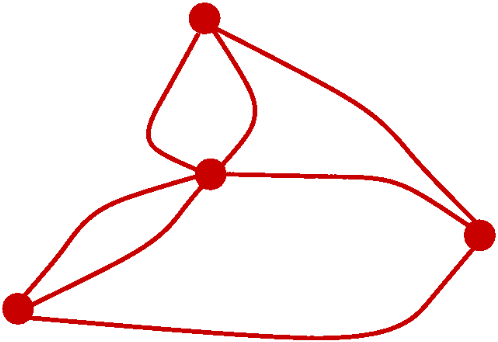
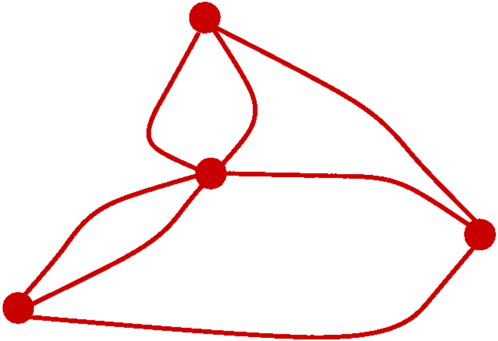


The required path exists if each vertex has even degree.



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Theorem (Euler 1736)

A **connected** graph has a path traversing each edge exactly once if, and only if, exactly zero or two vertices have odd degree.

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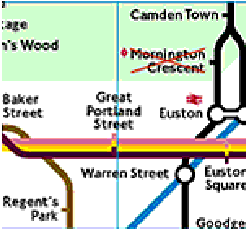

A topological map of the London underground.

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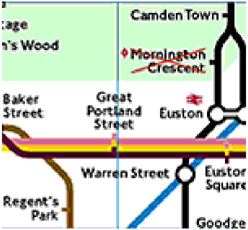

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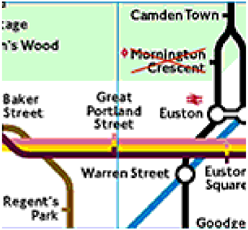

A topological map of the London underground.

The number of stops on the most direct route from Euston Square to Baker Street is a topological property of the London underground. The distance from Euston Square to Baker Street is **not**.

A doughnut has the same topological properties as a coffee mug.

A doughnut has different topological properties to a Starbucks cup.

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It has the topological properties of a berliner

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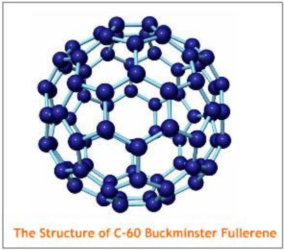
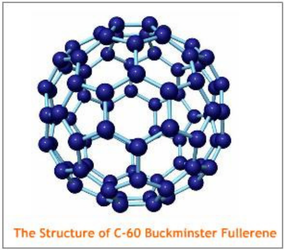
But what are these properties and what are they good for?


The Structure of C-60 Buckminster Fullerene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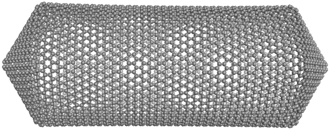

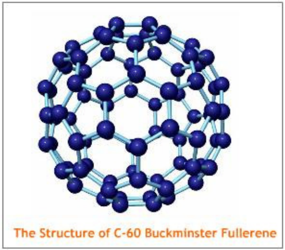
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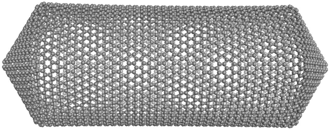

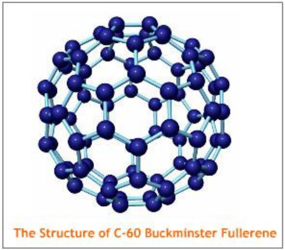

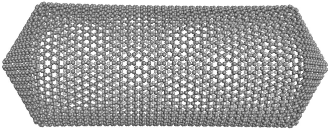

Fullerenes

What can be said about the number of pentagons in a fulleren?

Let's investigate this by first designing our own system of Villages and Expressways on Mars (or even on Toroidal Earth).

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- Expressways can cross only at villages.
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For our systems we should count:

$$\begin{aligned} V &= \text{number of villages} \\ E &= \text{number of expressways} \\ F &= \text{number of fields (regions enclosed by expressways)} \end{aligned}$$

We discover:

On a sphere	On a torus
V-E+F=2	V-E+F=0

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Euler gave a proof of $V - E + F = 2$ for the sphere.

His proof was not complete as it also (incorrectly) yields $V - E + F = 2$ on the torus!

Suppose
$P =$ number of pentagons and
$H =$ number of hexagons
on a soccer ball.

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This holds for **any** spherical fullerene!

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Exercise: How many pentagons in a toroidal fullerene?

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$$\chi(X) = V - E + F - F_3 + F_4 - F_5 + \cdots$$

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Around 1900 Poincaré proved

$$\chi(X) = \chi(Y)$$

if X can be continuously deformed into Y.

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Topology has flourished since 1900, with many topologists receiving the Fields Medal: Serre (1954), Thom (1958), Milnor (1962), Atiyah (1966), Smale (1966), Novikov (1970), Mumford (1974), Quillen (1978), Thurston (1982), Freedman (1986), Donaldson (1986), Jones (1990), Voevodsky (2002), Perelman (2006) , ...

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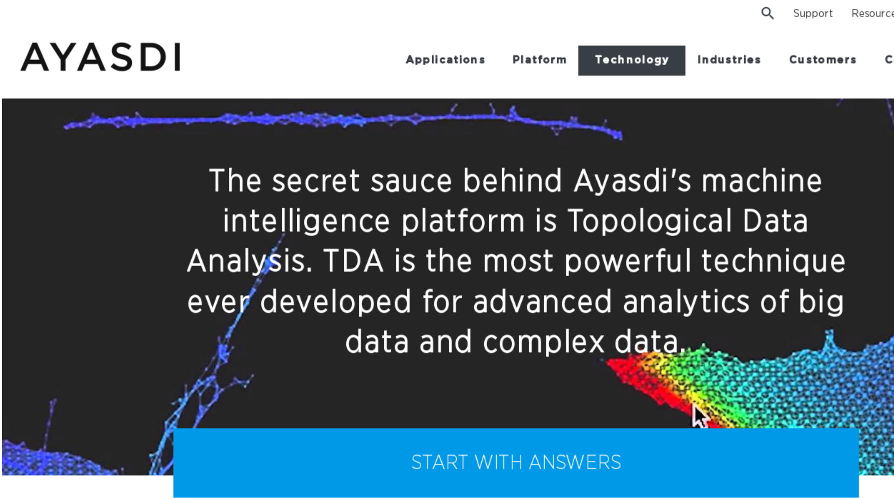
Applied Topology has emerged over the last decade as an extremely active area of research.


Support
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AYASDI
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Customers
The secret sauce behind Ayasdi's machine intelligence platform is Topological Data Analysis. TDA is the most powerful technique ever developed for advanced analytics of big data and complex data.
START WITH ANSWERS

For a cellular space X the Euler characteristic decomposes as

$$\chi(X) = \beta_0(X) - \beta_1(X) + \beta_2(X) - \beta_3(X) + \cdots$$

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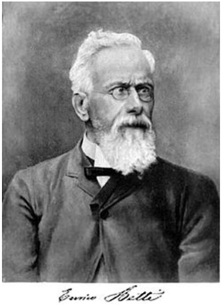

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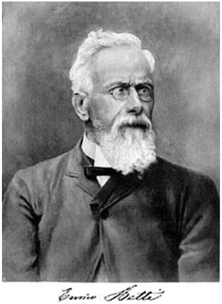

β_n is called a Betti number

$\beta_0(X) =$ number of connected components of X.
$\beta_n(X) =$ number of "n-dimensional holes" in X.

Fundamental problem from applied topology

Given a set S of points randomly sampled from an unknown manifold M, what can we infer about the topology of M?

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For instance, $S \subset M \subset \mathbb{E}^2$.

One approach to the problem

Repeatedly "thicken" the set S to produce a sequence of inclusions

$$S = S_1 \subset S_2 \subset S_3 \subset \cdots$$

and then search for "persistent" topological features in the sequence.

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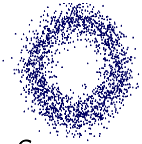

S_1

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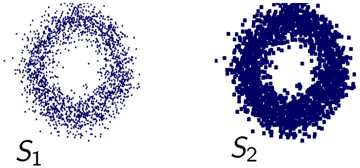

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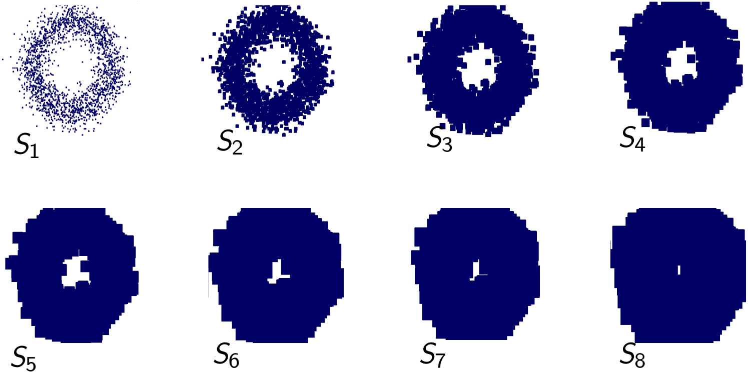

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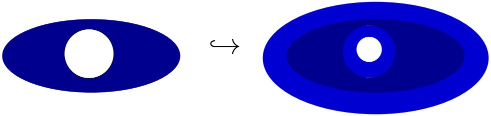
β_0	478	32	9	2	1	1	1	1
β_1	0	115	18	4	1	1	1	1

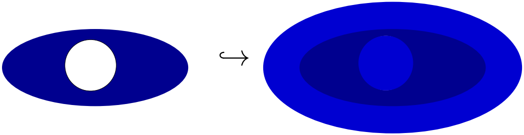
These numbers are consistent with the sample coming from some region with the homotopy type of a circle.

During an inclusion $S_s \hookrightarrow S_t$ holes can

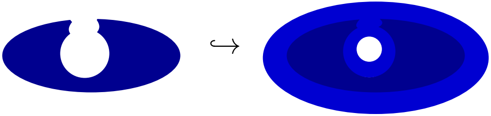
persist



die



be born



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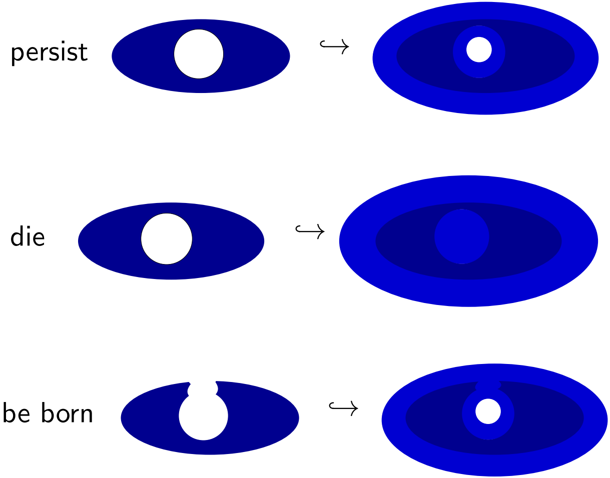

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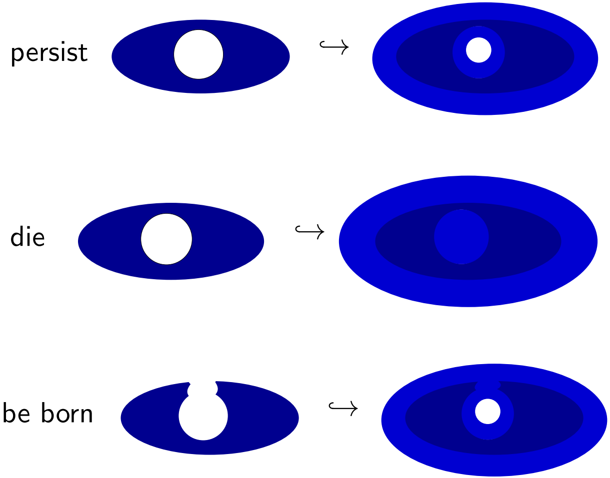

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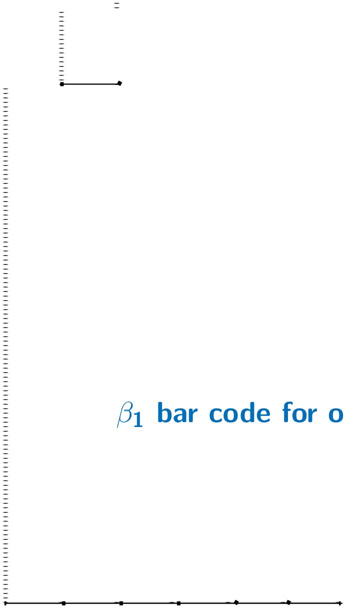
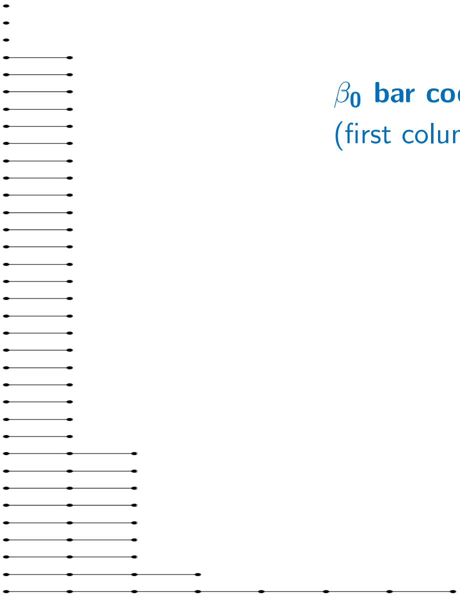
β_0^{st} = number of path components in S_s that persist to S_t

β_1 bar code for our example





β_0 bar code for our example

(first column cropped)

Topological data analysis can be tried on any sample of data for which there is some measure of distance between data points.

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The Mercury News
June 2013

"Muthu Alaggapan, an Ayasdi data scientist and also a Stanford Med School student shared his revolutionary approach to sports analytics using TDA which has gained him recognition as Forbes 30 Under 30 in Sports and a recent demo to President Obama. Muthu shared how using TDA he was able to discover that based on the data there are really 10 positions in basketball, not 5, and how these insights can lead to optimize team performance, player performance, real-time game decisions, and how teams are put together."

THANK YOU!