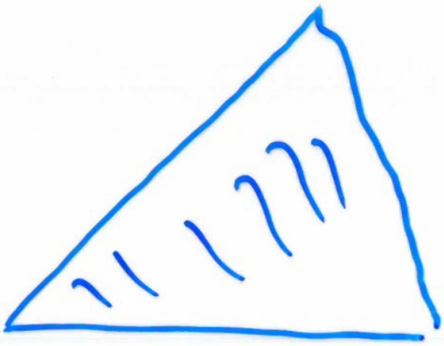
A space-filling curve (Peano Curve)

Let Δ be an equilateral triangular region in $\mathbb{E}^2$ of side 1

$\Delta =$  $\subseteq \mathbb{E}^2$

Theorem (Peano) There exists a surjective continuous function

$$f : [0,1] \longrightarrow \Delta .$$

Proof We first construct a sequence of continuous functions

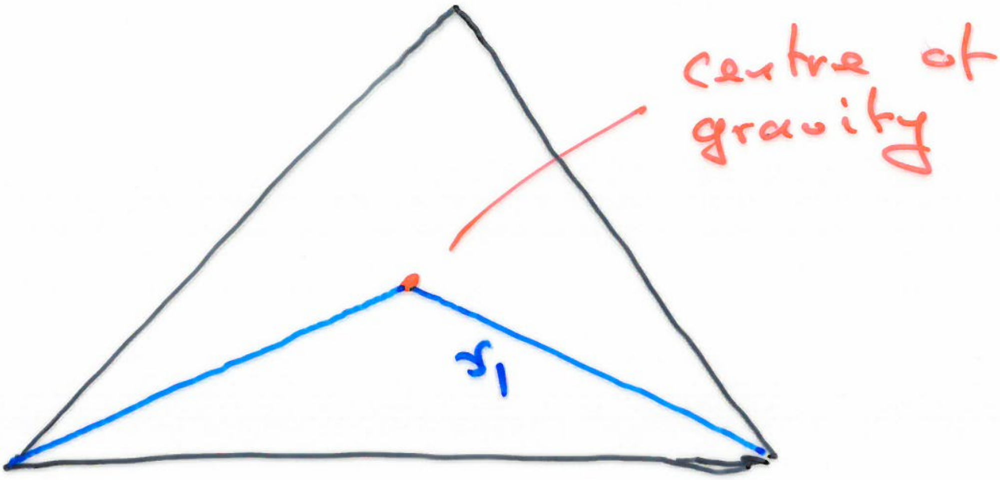
$f_1 : [0,1] \longrightarrow \Delta,$

$f_2 : [0,1] \longrightarrow \Delta,$

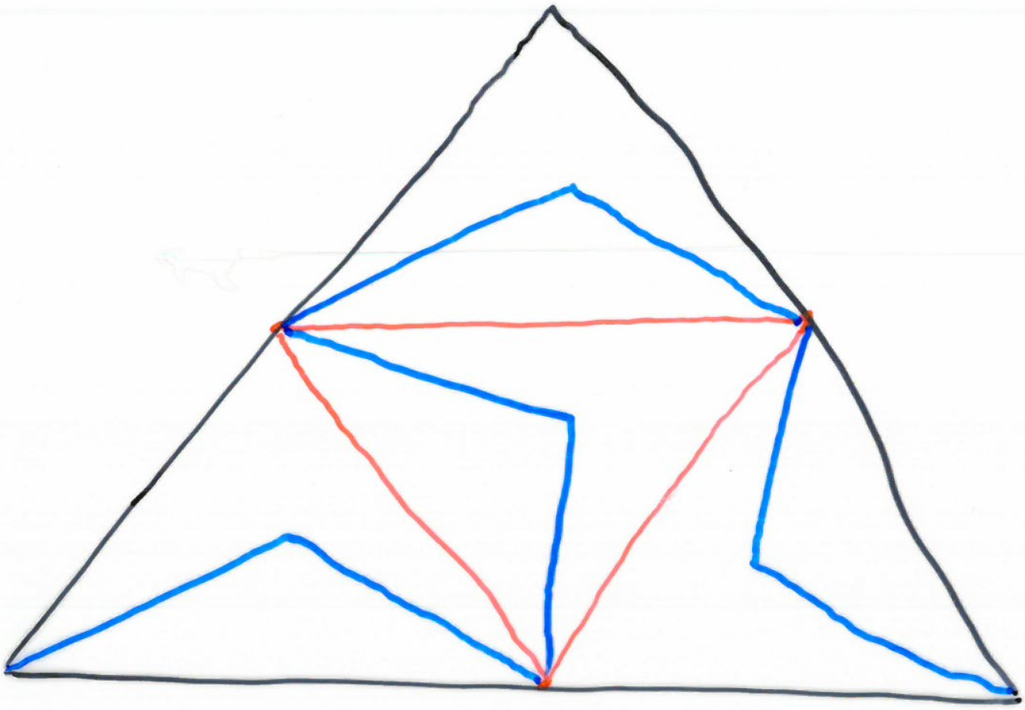
$f_3 : [0,1] \longrightarrow \Delta,$

$\vdots$

$f_1(t)$

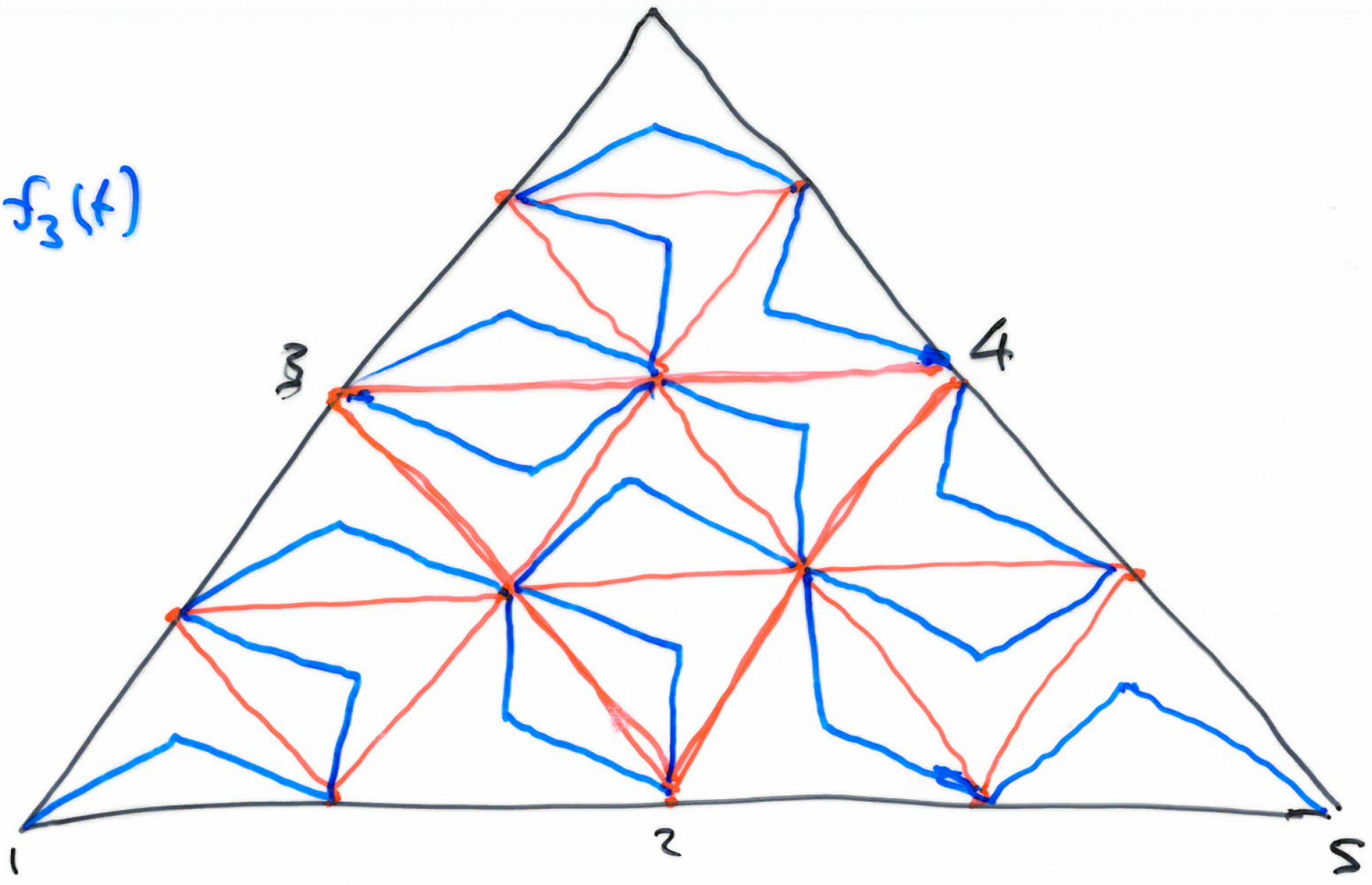


$f_2(t)$



f_2 constructed from four shrunk versions of f_1.

$f_3(t)$



For $f_n : [0,1] \to \Delta$ we subdivide Δ into 4^{n-1} red triangles, and the image of $f_n : [0,1] \to \Delta$ inside each red triangle looks just like f_1.

To complete the proof of the theorem we:

1) Define f to be the limit of $f_1, f_2, f_3, \dots$ (we need to check that this limit exists)

2) Need to prove that f is onto ("compactness" is used here).

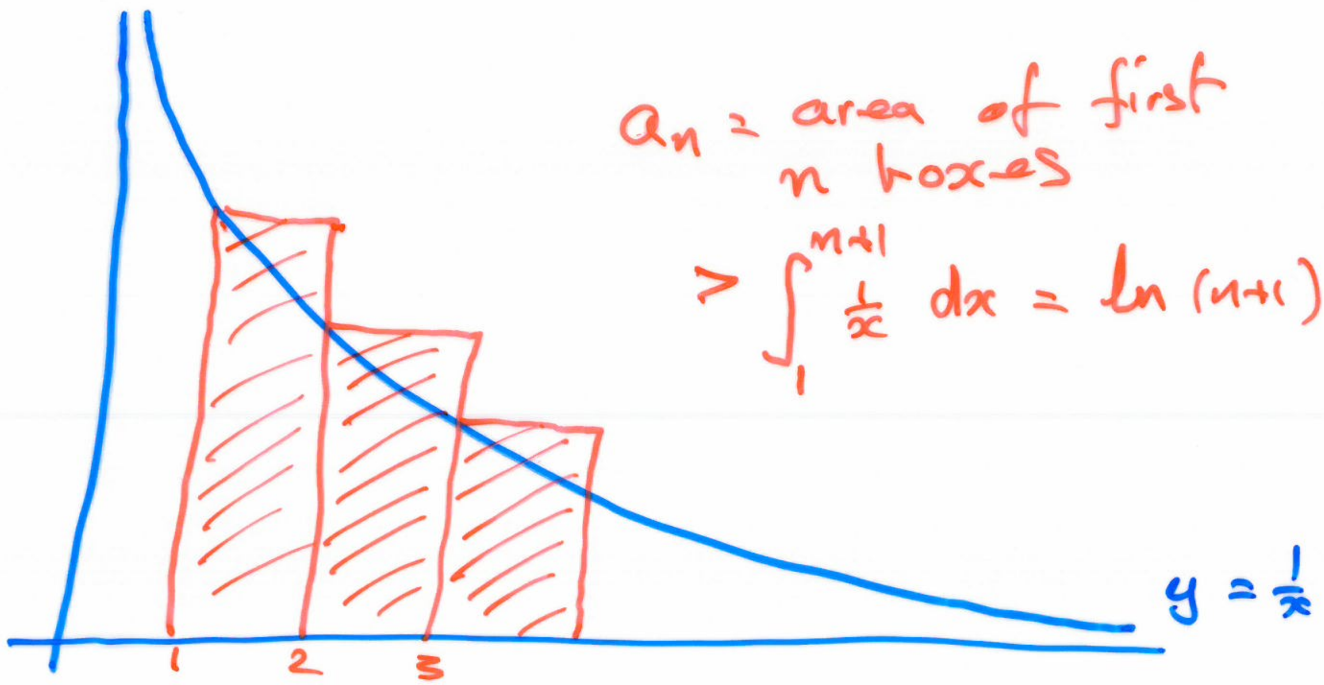
Some basics on limits.

Consider

$a_1 = 1,\ a_2 = 1\frac{1}{2},\ a_3 = 1\frac{5}{6}, \dots \quad a_n = a_{n-1} + \frac{1}{n}$

So $\lim_{n\to\infty} a_n - a_{n-1} = \lim_{n\to\infty} \frac{1}{n} = 0$.

$$\lim_{n\to\infty} a_n = \text{does not exist}$$



a_n = area of first n boxes

$$> \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

Defn A sequence of points $a_1, a_2, \ldots$ in $\mathbb{E}^k$ is said to be a Cauchy sequence if, for any $\varepsilon > 0$ there is an N such that

$$\| a_m - a_n \| < \varepsilon$$

for all $m, n \geq N$.

Theorem Any Cauchy sequence $a_1, a_2, \dots$ in $\mathbb{E}^k$ has a limit $\lim_{n\to\infty} a_n$.

Return to Peano

we've constructed functions $f_1 : [0,1] \to \Delta$, $f_2 : [0,1] \to \Delta, \dots$.

Fix $t \in [0,1]$.

For any $m > n$ we have that $f_m(t)$ and $f_n(t)$ both lie in a single equilateral triangle of side $\frac{1}{2^n}$.

So the sequence

$f_1(t), f_2(t), f_3(t), f_4(t), \dots$

is a Cauchy sequence. The sequence thus has a limit which we denote $f(t)$.

Furthermore, if t is close to t' then $f(t)$ is close to $f(t')$. "Thus" $f : [0,1] \to \Delta$ is continuous.

It remains to prove that $f(t)$ is surjective. (See book once we've covered "compactness".)