Continuity

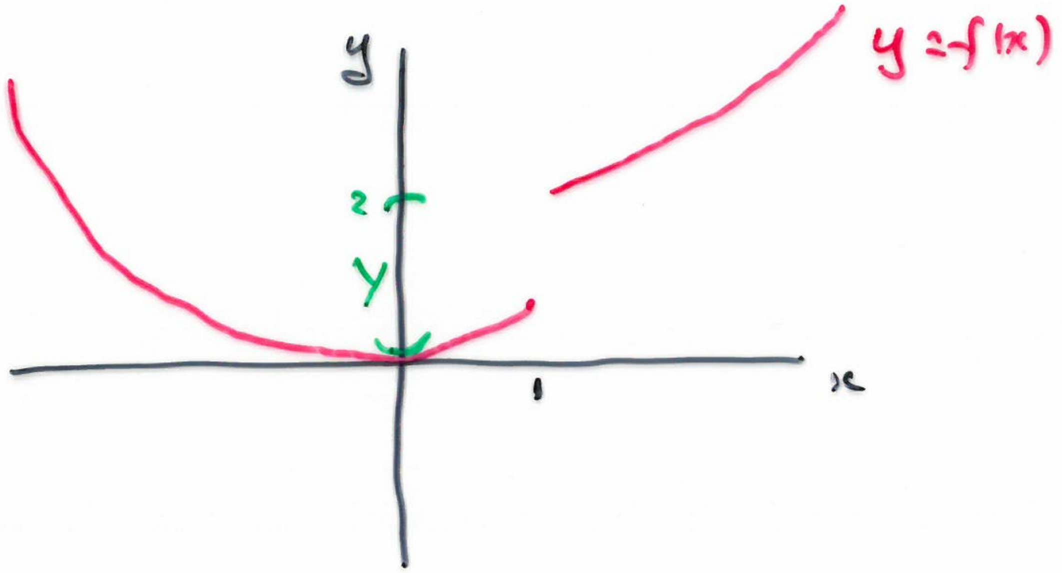
Defn Let X, Y be topological spaces. A function $f : X \to Y$ is <u>continuous</u> if the inverse image

$$f^{-1}(U) = \{ x \in X : f(x) \in U \}$$

of every open set U in Y is open in X.

Example $X = \mathbb{E}^1$, $Y = \mathbb{E}^1$.
Consider $f : \mathbb{E}^1 \to \mathbb{E}$ given by

$$f(x) = \begin{cases} x^2 & , \ x \leq 1 \\ x^2+1 & , \ x > 1 \end{cases}$$



Consider $\mathcal{U} = (0, 2) \subset Y = \mathbb{E}^1$

Then

$$f^{-1}(\mathcal{U}) = (-\sqrt{2}, 1]$$

is <u>not</u> open in $X = \mathbb{E}^1$. Hence

f is <u>not</u> continuous.

<u>Example</u> $X = (-\infty, 1) \cup (1, \infty)$

$Y = \mathbb{R}$

$g : X \to Y$ given by

$$g(x) = \begin{cases} x^2 & , x < 1 \\ x^2 + 1 & , x > 1 \end{cases}$$

This function **is** continuous.

Major Definition: A continuous function of topological spaces $f: X \to Y$ is a **homeomorphism** if there exists a continuous function $g: Y \to X$ such that

$$g(f(x)) = x \quad \text{for all } x \in X$$

and

$$f(g(y)) = y \quad \text{for all } y \in Y.$$

If $f: X \to Y$ is a homeomorphism then we say that X is **homeomorphic** to Y.

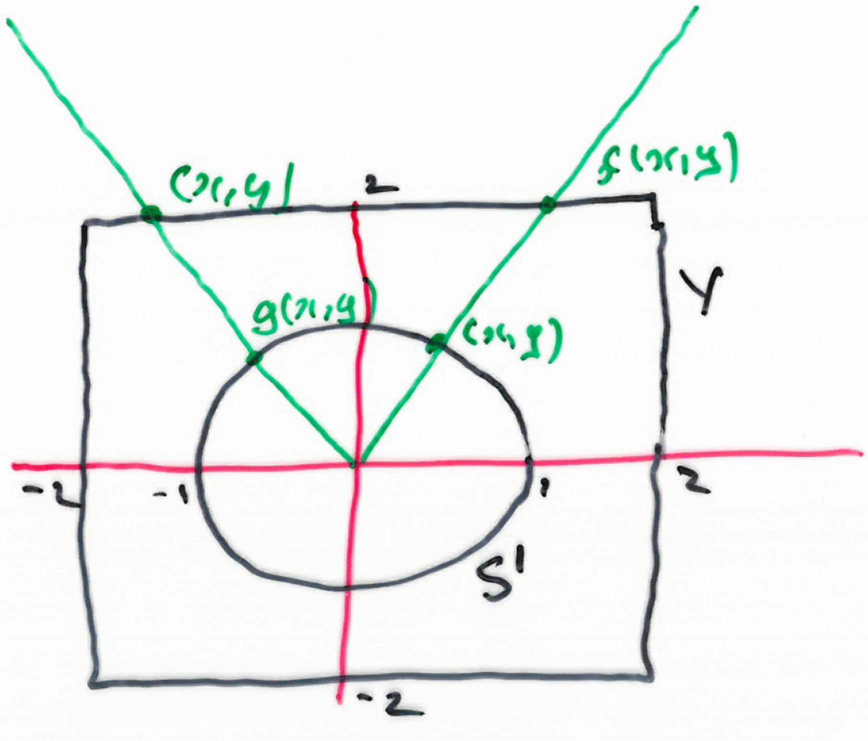
Example The unit circle

$$S^1 = \{ (x,y) \in \mathbb{E}^2 : x^2+y^2=1 \}$$

is homeomorphic to the square Y of side $\cancel{2}4$,

$$Y = \{ (x,y) \in \mathbb{E}^2 : -2 \leq x, y \leq 2 \text{ and either } x \in \{-2,2\} \text{ or } y \in \{-2,2\} \}.$$

Proof



Consider $f : S^1 \to Y$, $(x,y) \longmapsto f(x,y)$

where $f(x,y)$ is the intersection of

the ray from the origin through point (x,y) and the space Y.

Consider $g: Y \to S^1$, $(x,y) \mapsto g(x,y)$ where $g(x,y)$ is the intersection of the ray through the origin and the point (x,y) with the space S^1.

Note that both f and g are continuous. Furthermore,

$$g(f(x,y)) = (x,y)$$

and

$$f(g(x,y)) = (x,y).$$

Hence the square and the circle are homeomorphic.

$\boxtimes$

Example

A doughnut is homeomorphic to a coffee cup.

Proposition If $f : X \to Y$ and $h : Y \to Z$ are continuous functions then their composite

$$h \circ f : X \to Z, \quad x \mapsto h(f(x))$$

is continuous.

Proof Let $U \subset Z$ be any open set in Z. Then $h^{-1}(U) \subset Y$ is open in Y since h is continuous.

But $f^{-1}(h^{-1}(U)) \subset X$ is open in X because $h^{-1}(U)$ is open in Y and f is continuous.

Note that

$$(h \circ f)^{-1}(U) = f^{-1}(h^{-1}(U)).$$

So $h \circ f$ is continuous because the inverse image of any open set $U \subset Z$ is open in X. $\boxtimes$