Definition (Riesz [1909], Hausdorff [1914])

A **topological space** consists of a set X and a collection $\mathcal{T}$ of subsets of X which we deem to be "open". The following axioms must hold.

T1) The union of any collection of open sets is open.

T2) The intersection of any finite collection of open sets is open.

T3) Both $\emptyset$ and X are open

<u>Definition</u> A topological space X is said to be <u>connected</u> if it can not be expressed as a union

$$X = U \cup V$$

where U, V are non-empty open subsets of X, and where

$$U \cap V = \emptyset .$$

<u>Example</u> Let

$X = \mathbb{R}^n = \{ (x_1, \dots, x_n) : x_i \in \mathbb{R} \}$.

Let T consist of those subsets $U \subseteq \mathbb{R}^n$ such that for any $x \in U$ we can find $\varepsilon > 0$ such that the open Euclidean ball $B^n(x, \varepsilon)$ lies entirely in U:

$$B^n(x, \varepsilon) \subseteq U .$$

Then X, T is a topological space

and it is connected.

Example Let $X = \mathbb{R}^n$. Let T consist of all subsets of X. Then X, T is a topological space. We call this topology the discrete topology. This space is not connected. For instance

$$U = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 > 0\}$$

$$V = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}$$

Then $X = U \cup V$, $U \cap V = \phi$ and U, V are open.

Example Let $X = \mathbb{R}^n$. Let T consist of just two open sets:

$$T = \{\phi, X\}.$$

Then X, T is a topological space. We call this topology the

trivial topology. This topological space is connected.

Example Let $X = \mathbb{Z}$. The cofinite topology on X has as open sets those subsets $U \subseteq X$ such that the complement $X \setminus U$ is finite. Also, the empty set $\emptyset$ is deemed to be open. This is a topological space. With this topology $\mathbb{Z}$ is connected.

Definition Let X be a set with topology T. Let $Y \subseteq X$ be a subset. In the subspace topology on Y a subset $U \subseteq Y$ is open if

$$U = Y \cap A$$

with A an open subset of X.

with this topology we call Y a topological subspace of X.

Example Consider $X = \mathbb{R}$ with the standard topology in which a set $U \subseteq \mathbb{R}$ is open if, for each $x \in U$, there is an $\varepsilon > 0$ with

$$(x - \varepsilon, x + \varepsilon) \subseteq U.$$

Consider the integers $\mathbb{Z} \subseteq \mathbb{R}$ with the subspace topology. Then the space $\mathbb{Z}$ is not connected because

$$U = \{ n \in \mathbb{Z} : n \geq 0 \}$$

$$V = \{ n \in \mathbb{Z} : n < 0 \}$$

are open in the subspace, $\mathbb{Z} = U \cup V$, $U \cap V = \emptyset$.