The <u>sphere</u>

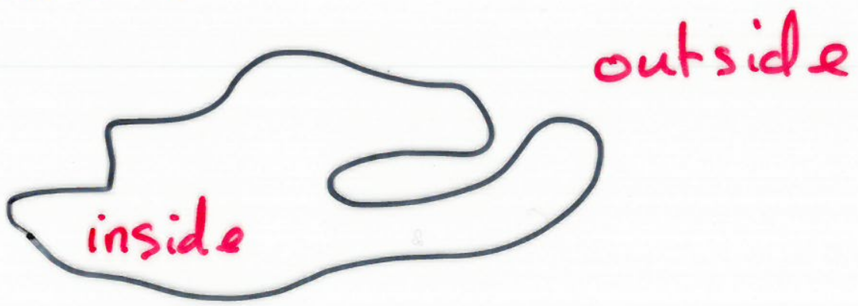
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$$

is a more precise notion than "the surface of Mars".

Our proof of the Euler charateristic formula

$$\chi(S^2) = 2$$

used the fact that any loop on the sphere with no self-intersections

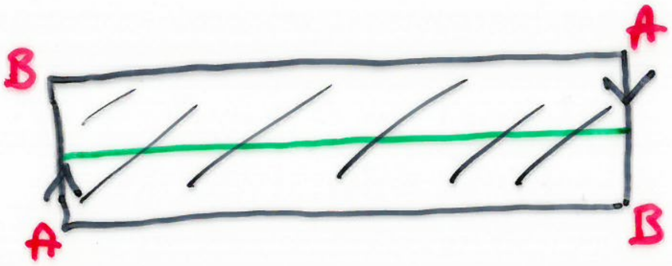


has an inside and an outside.

i.e. any such loop cuts the sphere into two regions.

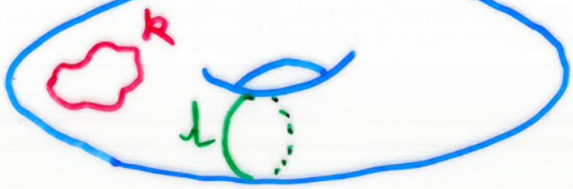
Q. Is this "fact" obvious?

Example Consider the Möbius strip.



Draw a loop around the centre of the strip. With paper, glue and scissors, check for yourself that this (green) loop does <u>not</u> cut the Möbius strip into two pieces.

Example Consider a torus (or



surface of a doughnut). Loop k cuts the torus into two pieces, but loop l does <u>not</u> !!

Example Is it obvious that the following loop in the plane $\mathbb{R}^2$ has an inside and an outside?

Topology offers a precise language and collection of techniques for studying such problems.

$$S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$$

is our notation for the circle.

This MA342 Module will enable you to understand the statement of the following result, and also its proof given in Armstrong's topology text.

Jordan Curve Theorem

Let $\alpha : S^1 \longrightarrow \mathbb{R}^2$ be any injective continuous function. Let $J \subseteq \mathbb{R}^2$ be the image of α. Then $\mathbb{R}^2 \setminus J$ has precisely two connected components both of which have frontier J.

Aim for next few lectures:

1) Explain underlined terms
2) Give the explanation using the

notion of a "continuous function between topological spaces"

$f: X \to Y$

3) Give some weird examples that suggest the theorem is not so obvious

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For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define the <u>Euclidean</u> <u>norm</u>

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

For $x, y \in \mathbb{R}^n$ we define the <u>Euclidean</u> <u>distance</u>

$$d(x,y) = \|x - y\|$$

We write $\mathbb{E}^n$ to denote the set $\mathbb{R}^n$ endowed with the Euclidean distance.

we define the n-sphere

$$S^n = \{ x \in \mathbb{E}^{n+1} : \| x \| = 1 \}$$

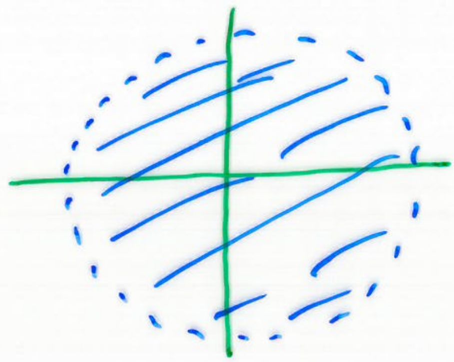
S^1 :  S^2 : 

For $x \in \mathbb{E}^n$ and for any real number $\varepsilon > 0$ we define the open ball centred at x of radius ε to be

$$B^n(x, \varepsilon) = \{ y \in \mathbb{E}^n : d(x,y) < \varepsilon \}$$

Examples

$B^2((0,0), 1)$



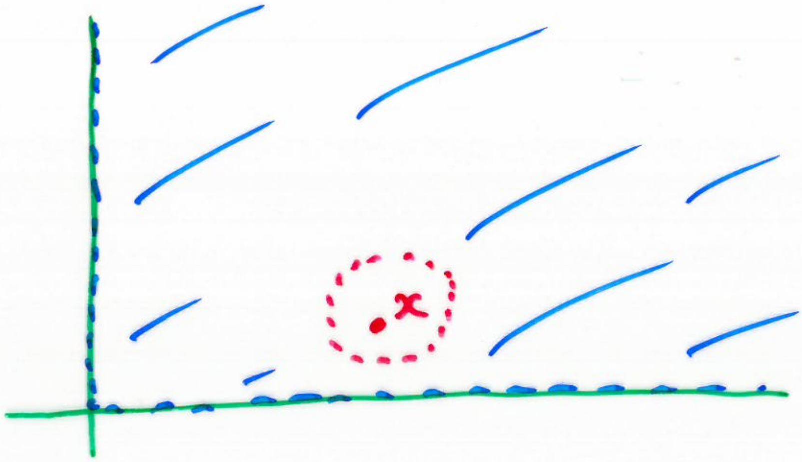
$B^1(0,1)$
$= (-1, 1)$

Defn A set $X \subseteq \mathbb{E}^n$ is said to be **open** if, for any $x \in X$, we can find some $\varepsilon > 0$ (where ε can depend on x) such that

$$B^n(x, \varepsilon) \subseteq X.$$

Example

Consider $X = \{(x,y) \in \mathbb{E}^2 : x > 0, y > 0\}$



This X is an open subset of $\mathbb{E}^2$.

Example Consider

$$X = \{x \in \mathbb{R} : 0 < x < 1\} = (0, 1).$$

This set X is an open subset of $\mathbb{R} = \mathbb{E}^1$.

Example S^2 is **not** an open subset of $\mathbb{E}^3$.

Three important facts

Fact 1 If $X_1, X_2, \dots$ is a collection of open sets in $\mathbb{E}^n$ then their union

$$X_1 \cup X_2 \cup \cdots$$

is an open set in $\mathbb{E}^n$.

Fact 2 If $X_1, X_2, \dots, X_n$ is a finite collection of open sets in $\mathbb{E}^n$ then their intersection

$$X_1 \cap X_2 \cap \dots \cap X_n$$

is an open set in $\mathbb{E}^n$.

Fact 3 The empty set $\emptyset$ and the set $X = \mathbb{E}^n$ are open subsets of $\mathbb{E}^n$.

__Remark__ Fact 2 does __not__ hold for some infinite collections.

Consider

$$X_i = (0, 1 + \tfrac{1}{i}) \subseteq \mathbb{E}^1$$

for $i = 1, 2, 3, \ldots$.

Then

$$\bigcap_{i \geq 1}^{\infty} X_i = (0, 1]$$

is __not__ open in $\mathbb{E}^1$.