Applied Topology?

(Computing homotopy 0-types, 1-types and 2-types)

Graham Ellis
NUI Galway

Homotopy 0-types

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and is a homotopy invariant: $X \simeq Y \Rightarrow \pi_0(X) = \pi_0(Y)$.

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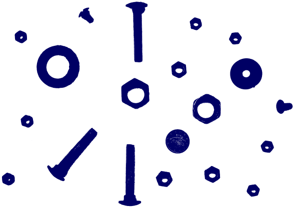
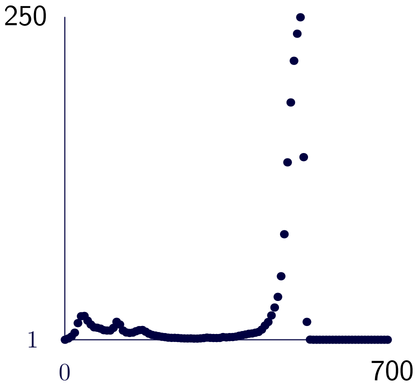
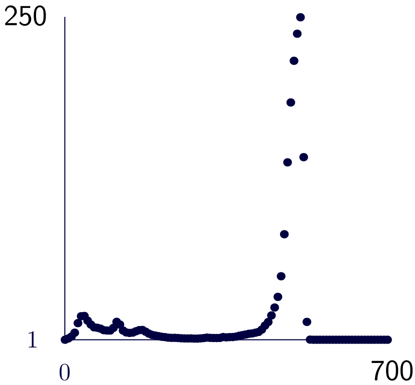
$$S_t = \{(x, y) \in \mathbb{R}^2 \ : \ ||f(x, y)|| \leq t\}$$





Plot of $|\pi_0(S_t)|$ as a function of t



Plot of $|\pi_0(S_t)|$ as a function of t

$t_1 < t_2 < \cdots < t_T$ implies $S_{t_1} \subset S_{t_2} \subset \cdots \subset S_{t_T}$

$$\beta_0^{t,t'} = |\mathsf{image}(\ \pi_0(S_t) \to \pi_0(S_{t'})\ )| \text{ for } t \leq t'$$

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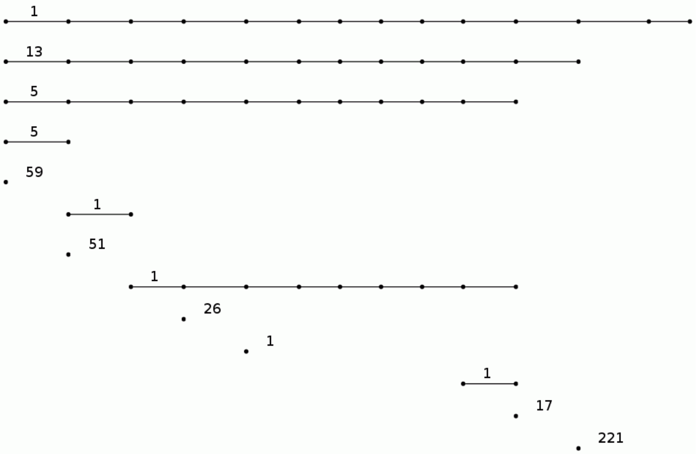

r lines from column t to column t' if $\beta_0^{t,t'} = r$

Homotopy 1-types

The fundamental group

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$H_1(X, \mathbb{Q}) = (\pi_1(X, x_0))_{ab} \otimes \mathbb{Q}$

$\beta_1(X) = \dim(\ H_1(X, \mathbb{Q})\ ) =$ 'number of 1-dimensional holes in X'

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Second Toy Example

S sampled from $M \subset \mathbb{R}^2$

Repeatedly "thicken" the data S to produce a sequence of inclusions

$$S = S_1 \subset S_2 \subset S_3 \subset \cdots$$

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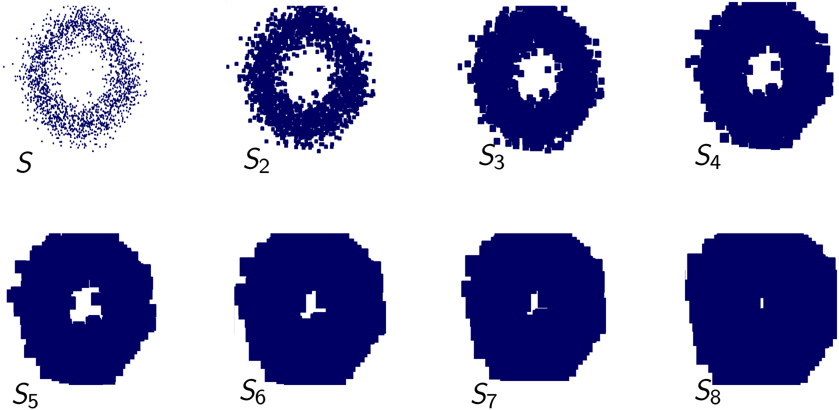

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	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
β_1	0	115	18	4	1	1	1	1

These numbers are consistent with the sample coming from some region with the homotopy type of a circle.

$$\beta_1^{t,t'} = \mathrm{rank}(\ H_1(S_t, \mathbb{Q}) \to H_1(S_{t'}, \mathbb{Q})\ ), \quad t \leq t'$$

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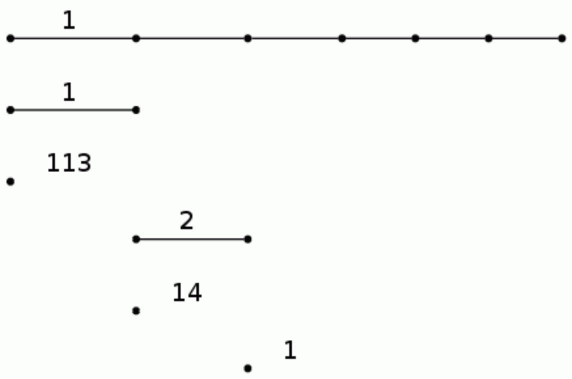
β_1 bar code for our example



Third Toy Example $v_1, v_2, \ldots, v_{72} \subset \mathbb{R}^{262144}$

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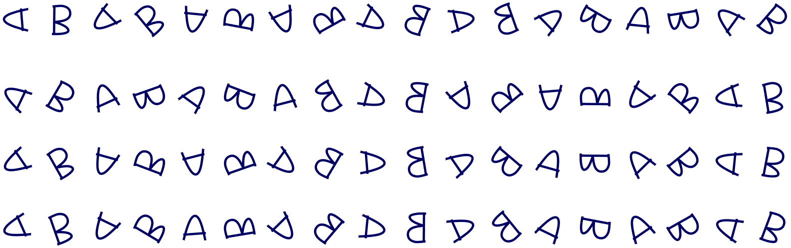
Toy data points from

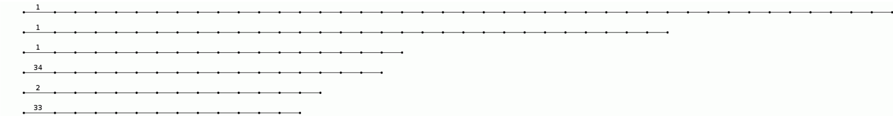


Fix a sequence of real numbers $0 < t_1 < t_2 < \cdots < t_T$.

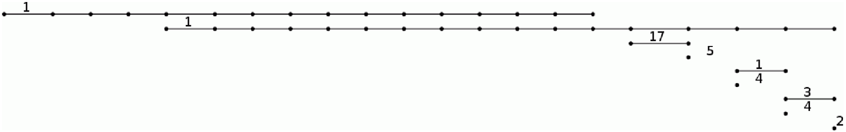
The **Rips simplicial complex** X_t has with

- vertex set $V = \{v_1, \ldots, v_{72}\}$.
- n-simplices the subsets $\sigma \subseteq V$ with $n+1$ vertices and $||v - v'|| \leq t$ for all $v, v' \in \sigma$.

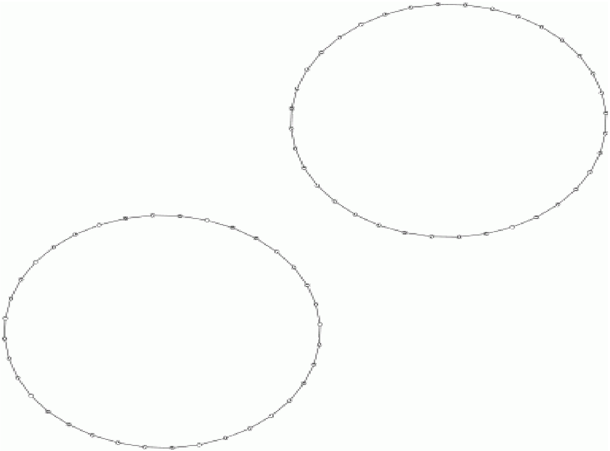
Persistent β_0 for $C_*(X_*)$:



Persistent β_1 for $C_*(X_*)$:



Data Model: A homotopy retract $Y \subset X_{20}$



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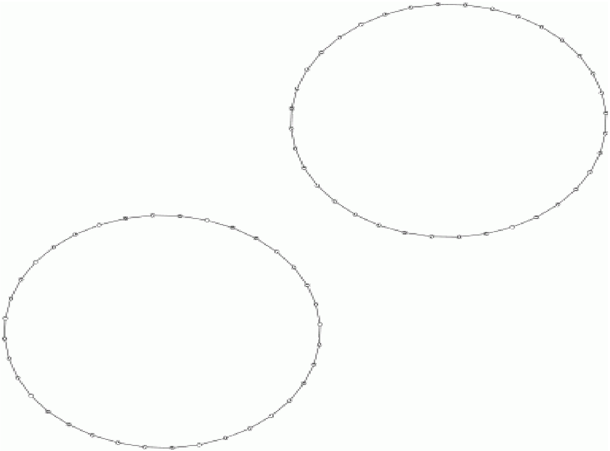
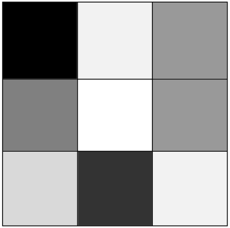


$$\mathbf{Y} \simeq \mathbf{S^1} \sqcup \mathbf{S^1}$$

Fourth Example

Mumford, Lee, Pedersen: 8 000 000 random high-contrast 3×3 patches in $\mathbb{E}^9$.



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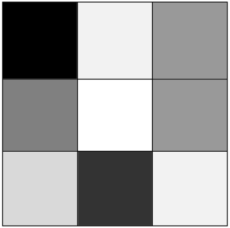
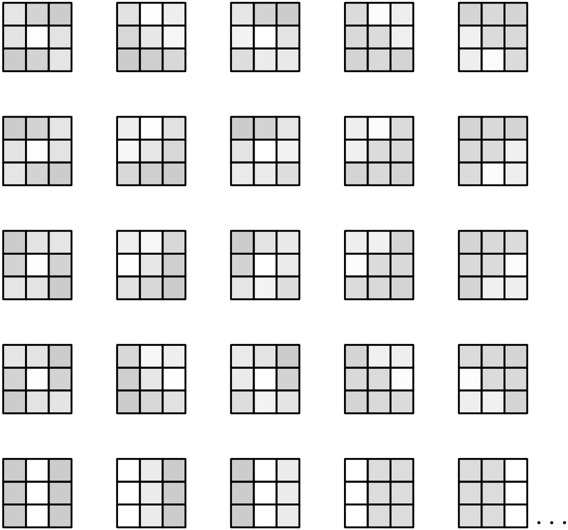
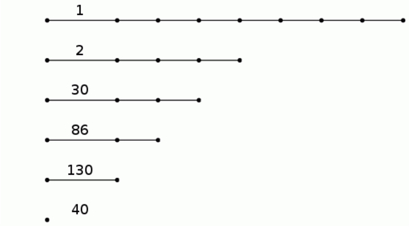


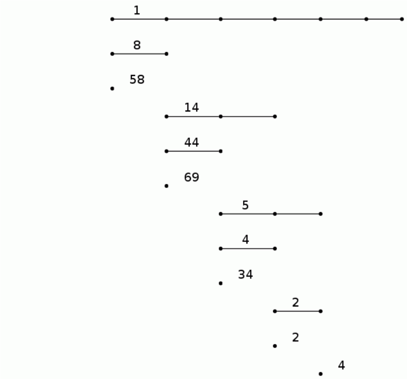
Carlsson, Ishkanov, de Silva, Zomorodian: performed topological analysis of the data.

We take a small synthetic data set $v_1, v_2, \ldots, v_{256} \subset \mathbb{R}^9$ based on



β_0 and β_1 bar codes for the 256 patches:





Data Model: A homotopy retract $Y \subset X_9$ with

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- and just three critical cells e^0, e^1_1, e^1_2, e^2.

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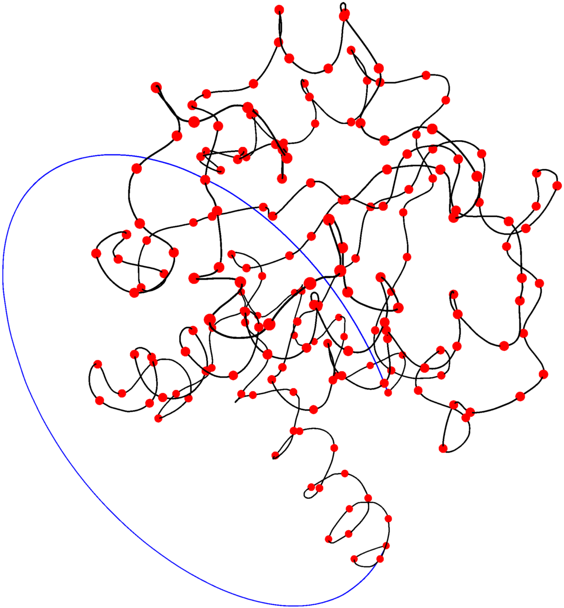
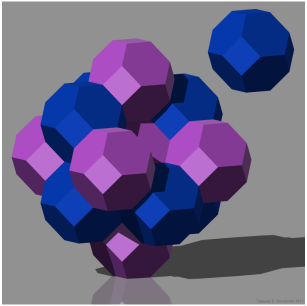
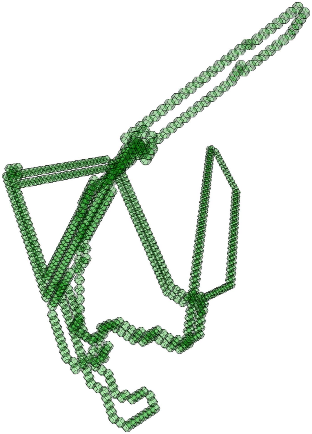
Caveat: [Kan-Thurston] For any space X there is a map $K(G,1) \to X$ inducing $H_*(K(G,1)) \cong H_*(X)$.

Fifth Example H. Sapiens 1v2x protein




Tadeusz E. Dorozinski 2011



Proposition: *The alpha carbon atoms of the Thermus Thermophilus protein determine a knot K with peripheral system*

$$\begin{array}{rcl} \pi_1(\partial K) \cong \langle a, b | aba^{-1}b^{-1} \rangle & \to & \pi_1(\mathbb{R}^3 \setminus K) \cong \langle x, y | xyx = yxy \rangle \\ a & \mapsto & x^{-2}yx^2y \\ b & \mapsto & x \end{array}.$$

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```
gap> K:=ReadPDBfile("1V2X.pdb");
Pure permutahedral complex of dimension 3

gap> Y:=RegularCWComplex(PureComplexComplement(K));;
Regular CW-complex of dimension 3

gap> i:=Boundary(Y);
Map of regular CW-complexes

gap> phi:=FundamentalGroup(i,22495);
[ f1, f2 ] -> [ f1^-3*f2*f1^2*f2*f1, f1 ]
```

Proposition (Brendel, E, Juda, Mrozek)

The abelian invariants of subgroups $G \leq \pi_1(\mathbb{R}^3 \setminus K)$ *of index* ≤ 7 *distinguish between the* 46972 *ambient isotopy classes of prime knots with fourteen or fewer crossings, up to chirality.*

Homotopy 2-types

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How do we represent homotopy 2-types?

A CW complex X with 1-skeleton X^1 yields a fibration

$$F \rightarrowtail \overline{X}^1 \twoheadrightarrow X$$

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satisfies

- $\partial({}^g m) = gmg^{-1}$,
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This structure $\Pi(X)$ is a **free crossed module** if $X = X^2$.

For $\mathcal{P} = \langle x_1, \ldots, x_d \;:\; r_1, \ldots, r_m \rangle$ and $X = K(\mathcal{P})$

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- $G =$ free group on the x_i
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$$ {}^g\mathbf{r}_i \, {}^h\mathbf{r}_j \, {}^g\mathbf{r}_i^{-1} = {}^{g r_i g^{-1} h}\,\mathbf{r}_j $$

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$$\mathcal{P} = \langle x_1, \ldots, x_d \;:\; r_1, \ldots, r_m \;:\; s_1, \ldots, s_n \rangle$$

with

- $r_i \in G =$ free group on x_i;
- $s_i \in \ker(\partial \colon M \to G)$ where ∂ is the free crossed module of $\langle x_1, \ldots, x_d \;:\; r_1, \ldots, r_m \rangle$.

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Theorem (Whitehead)

Every $\mathcal{P}$ determines, and is determined by, the cell structure on a 3-dimensional CW complex X.

$$\mathcal{P} = \left\langle \; x, y, z \; : \; [y,z],\; [x,z],\; [y,x] \; : \; \begin{array}{l} \\ [\mathbf{z},\mathbf{y}] \; {}^{y}x[\mathbf{z},\mathbf{y}]^{-1} \, [\mathbf{y},\mathbf{x}] \; {}^{x}z[\mathbf{y},\mathbf{x}]^{-1} \, [\mathbf{x},\mathbf{z}] \; {}^{z}y[\mathbf{x},\mathbf{z}]^{-1} \end{array} \; \right\rangle$$

where $[y, z] := yzy^{-1}z^{-1}$ and ${}^{y}x := yxy^{-1}$ and $\partial([\mathbf{z},\mathbf{y}]) = [z, y]$.

$$\mathcal{P} = \langle \ \ x, y, z \ : \ [y,z], \ [x,z], \ [y,x] \ : \ \mathbf{[z,y]} \ {}^{y}x \mathbf{[z,y]}^{-1} \mathbf{[y,x]} \ {}^{x}z \mathbf{[y,x]}^{-1} \mathbf{[x,z]} \ {}^{z}y \mathbf{[x,z]}^{-1} \ \ \rangle$$

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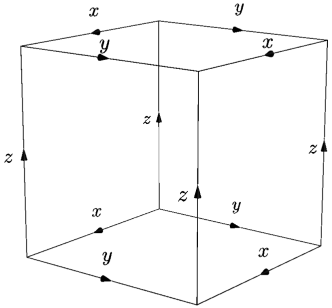

Corrsponding CW complex is $S^1 \times S^1 \times S^1$ where $S^1 = e^0 \cup e^1$.

Theorem (Whitehead)

$$\left\{ \begin{array}{l} \textit{Quasi equivalence} \\ \textit{classes of crossed} \\ \textit{modules} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \textit{homotopy classes} \\ \textit{of homotopy 2-} \\ \textit{types} \end{array} \right\}$$

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Given a free presentation for the crossed module $\Pi(X)$ and given any finite crossed module C, the set of homotopy classes of morphisms

$$[\Pi(X), C] = \{\Pi(X) \to C\}/\simeq$$

can be computed and is a homotopy invariant of X.

A morphism of crossed modules induces a group homomorphism on

$$\pi_1 = G/\partial M \tag{1}$$

and a π_1-module homomorphism on

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Quasi equivalence is the equivalence relation on crossed modules generated by morphisms where (1) and (2) are isomorphisms.

$$\partial \to \partial_1 \leftarrow \partial_2 \to \cdots \to \partial_n \leftarrow \partial'$$

The **order** of homotopy 2-type X is the least value of $m = |M||G|$ for a representative crossed module $M \stackrel{\partial}{\to} G$.

Proposition (E, Le)

The homotopy 2-types of order m are classified up to homotopy for $m \leq 127$, $m \neq 32, 64, 81, 96$ and are distributed with GAP.

$\partial\colon Q \to Aut(Q)$

```
gap> G2:=AutoCrossedModule(DihedralGroup(216));;

gap> Size(G2);
839808

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[ 72, 68 ]
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gap> Homology(G,5);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 18 ]
```
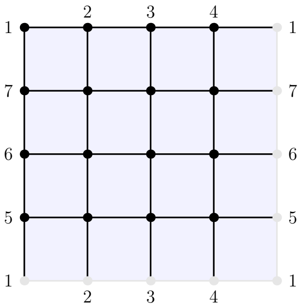
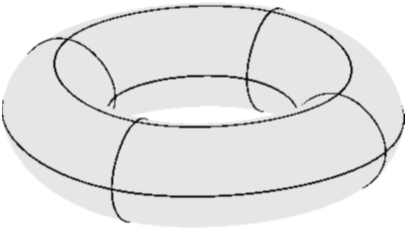
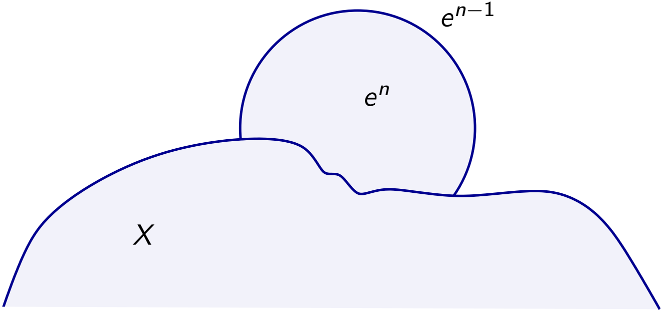

Computations?

A finite regular CW complex is readily stored on a computer.

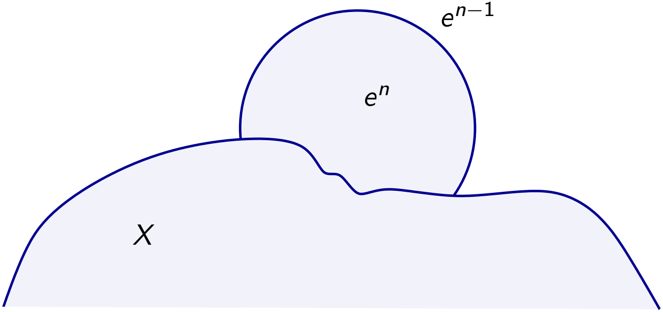




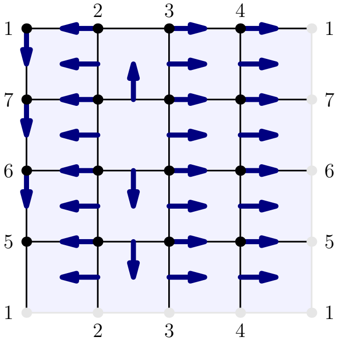
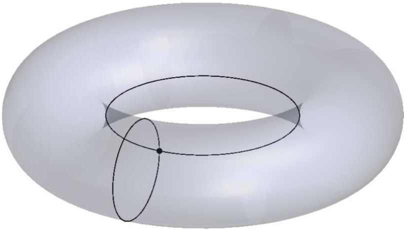
A **simple homotopy collapse** $X \cup e^n \cup e^{n-1} \searrow X$

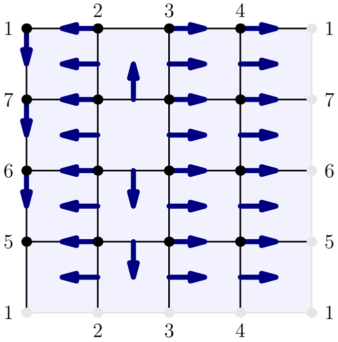


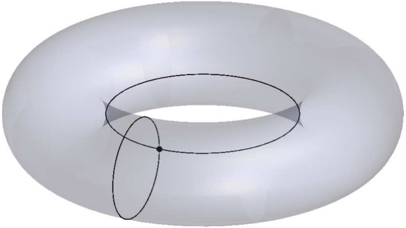
A **simple homotopy collapse** $\quad X \cup e^n \cup e^{n-1} \searrow X$



is stored in computer as a pair (e^{n-1}, e^n) or arrow $e^{n-1} \longrightarrow e^n$.


2 3 4
1 1
7 7
6 6
5 5
1 1
2 3 4






A **discrete vector field** is a collection of arrows $e^{n-1} \longrightarrow e^n$ with e^{n-1} in the boundary of e^n and with any cell involved in at most one arrow. It is **admissible** if there is no chain

$$\cdots (e_1^{n-1}, e_1^n), (e_2^{n-1}, e_2^n), (e_3^{n-1}, e_3^n), \cdots$$

with each e_{i+1}^{n-1} in the boundary of e_i^n and with infinitely many (not necessarily distinct) terms to the right.

A regular CW complex X with admissible discrete vector field represents a homotopy equivalence

$$X \xrightarrow{\simeq} Y$$

where Y is a CW complex whose cells correspond to those of X that are neither source nor target of any arrow.

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An admissible discrete vector field on X^2 with just one critical 0-cell represents a presentation for $\pi_1 X$.

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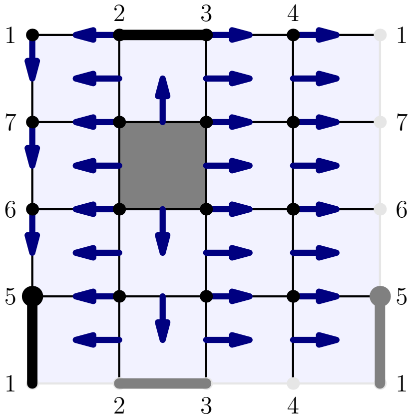

$$\pi_1 X = \langle x, y \mid xyx^{-1}y^{-1} \rangle$$