<u>Recall</u> A <u>game</u> involves

- n players
- a set S_i of strategies for player i
- a payoff function

$$\nu_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$

for player i, $1 \le i \le n$.

<u>Example</u> $n = 2$

$S_1 = \{H, T\}$, $S_2 = \{H, T\}$

$\nu_1(H,H) = 1$ $\nu_2(H,H) = -1$	$\nu_1(H,T) = -1$ $\nu_2(H,T) = 1$
$\nu_1(T,H) = -1$ $\nu_2(T,H) = 1$	$\nu_1(T,T) = 1$ $\nu_2(T,T) = -1$

<u>Recall</u> A <u>mixed strategy</u> is a choice of probabilities

$$p_{i,s}$$

for $1 \le i \le n$, $s \in S_i$ satisfying

$$p_{i,s} \ge 0, \quad \sum_{s \in S_i} p_{i,s} = 1.$$

Here $p_{i,s}$ is the probability that player i plays strategy $s \in S_i$.

<u>Notation</u> Suppose $S_i = \{s_1, s_2, \ldots, s_k\}$, and set

$$\underline{p}_i = (p_{i,s_1}, p_{i,s_2}, \cdots, p_{i,s_k}) \in \mathbb{R}^{|S_i|}$$

Define the expected payoff for player i to be the function

$$\mathcal{E}_i : \mathbb{R}^{|S_1|+|S_2|+\cdots+|S_n|} \longrightarrow \mathbb{R}$$

by

$$\mathcal{E}_i(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) = E(\nu_i)$$

$$= \sum_{\substack{x_1 \in S_1 \\ x_2 \in S_2 \\ \vdots \\ x_n \in S_n}} p_{1x_1} p_{2x_2} \cdots p_{nx_n} \nu_i(x_1, x_2, \ldots, x_n)$$

<u>Example</u> For our 2-player game

$$S_1 = \{H, T\}, \quad S_2 = \{H, T\}$$

$$\Sigma_i(\underline{P}_1, \underline{P}_2) =$$

$$P_{1H} P_{2H} \mathcal{V}_i(H, H) + P_{1T} P_{2H} \mathcal{V}_i(T, H)$$

$$+ P_{1H} P_{2T} \mathcal{V}_i(H, T) + P_{1T} P_{2T} \mathcal{V}_i(T, T)$$

$$\Sigma_1(P_{1H}, P_{1T}, P_{2H}, P_{2H}) = P_{1H} P_{2H} - P_{1T} P_{2H}$$

$$- P_{1H} P_{2T} + P_{1T} P_{2T}$$

~~Theorem (Nash)~~

<u>Example</u> In our 2-player game an example of a mixed Nash equilibrium is the strategy

$$P_{1H} = \frac{1}{2} \quad P_{1T} = \frac{1}{2} \quad P_{2H} = \frac{1}{2} \quad P_{2T} = \frac{1}{2}$$

Theorem (Nash) For any game with finitely many players and finite strategy sets there exists at least one mixed Nash equilibrium.

Outline proof

Consider

$$C = \{ (\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) \} \subseteq \mathbb{R}^{|S_1| + |S_2| + \cdots + |S_n|}$$

where $\underline{p}_i \in \mathbb{R}^{|S_i|}$ is the probability distribution for player i, $1 \le i \le n$.

Now C is closed, bounded and convex, since $p_{i,s} \geq 0$, $\sum_{s \in S_i} p_{is} = 1$.

Thus any continuous function $f: C \to C$ has at least one fixed point (Brouwer's Theorem).

For a given $(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) \in C$ define $\underline{q}_i \in \mathbb{R}^{|S_i|}$ to be the probability distribution $\underline{q}$ that maximizes

$$\max_{\underline{q}} \; \mathcal{E}_i(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_{i-1}, \underline{q}, \underline{p}_{i+1}, \ldots, \underline{p}_n) \quad (*)$$

Now define $f : C \to C$ by

$$f(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) = (\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_n)$$

This function f has a fixed point. But a fixed point is a Nash equilibrium. $\blacksquare$

Slight problem: The quantity $(*)$ might achieve its max. value at more than one value of $\underline{q}$. So our f is not well-defined!

To solve this problem replace (*) by

$$\max_{\underline{q}} \left(\xi_i(\underline{p}_1, \dots, \underline{p}_{i-1}, \underline{q}, \underline{p}_{i+1}, \dots, \underline{p}_n) - \|\underline{p}_i - \underline{q}\|^2 \right) \qquad (**)$$