Game Theory

A **game** involves

- n players
- a set S_i of strategies for player i
- a payoff function
  $$v_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$
  for player i, $1 \leq i \leq n$.

Example:

2 players, Mary and John who want to go to either the cinema (C) or a soccer match (S) together.

payoff:

$v_1(C,C) = 2$ $v_2(C,C) = 1$	$v_1(C,S) = 0$ $v_2(C,S) = 0$
$v_1(S,C) = 0$ $v_2(S,C) = 0$	$v_1(S,S) = 1$ $v_2(S,S) = 2$

Example 2

2 players, each places a coin on the table. Player 1 wants coins to be the same. Player 2 wants coins to be different.

$$S_1 = \{H, T\}$$
$$S_2 = \{H, T\}$$

payoff:

$\nu_1(H,H) = 1$ $\nu_2(H,H) = -1$	$\nu_1(H,T) = -1$ $\nu_2(H,T) = 1$
$\nu_1(T,H) = -1$ $\nu_1(T,H) = 1$	$\nu_1(T,T) = 1$ $\nu_2(T,T) = -1$

In a pure strategy game each player decides, beforehand, on a strategy to play. A pure Nash equilibrium occurs if, having played the game, no

player benefits from unilaterally changing his/her choice of strategy.

Example 1 There are two pure Nash equilibria: both go to the Cinema, or both go to soccer.

Example 2 There is no pure Nash equilibrium in this game.

Example A beautiful mind:

$n = 4$

Each player has two strategies:

A = talk to the blonde

B = talk to the brunette

$S_i = \{A, B\} \qquad 1 \le i \le 4.$

$$\nu_i(x_1, x_2, x_3, x_4) = \begin{cases} 10 & \text{if } x_i = A,\ x_j = B \\ & \quad j \ne i \\ 5 & \text{if } x_i = B, \text{ at} \\ & \text{most one } x_j = A \\ & j \ne i \\ 0 & \text{if two or more} \\ & x_i\text{s equal } A. \end{cases}$$

The choice of strategies $x_i = B$ all $1 \le i \le 4$, which was given in the film, is not a pure Nash equilibrium.

A pure Nash equilibrium occurs if precisely one man talks to the blonde.

In a mixed strategy game player i decides on a probability $p_i(s)$ to play strategy $s \in S_i$.

so $p_i(s) \geq 0$ and $\sum_{s \in S_i} p_i(s) = 1$.

The i^{th} player wants to maximize the expected payoff

$$E(\nu_i(s_1, s_2, \dots, s_n)) =$$

$$\sum_{\substack{s_1 \in S_1 \\ s_2 \in S_2 \\ \vdots \\ s_n \in S_n}} p(s_1)\, p(s_2) \cdots p(s_n)\, \nu_i(s_1, s_2, \dots, s_n)$$

A mixed Nash equilibrium occurs if, having played the game, no player unilaterally benefits by changing his/her mixed strategy.

Theorem (J. Nash) In any game with finitely many players and finite pure strategy sets S_i, there exists at least one mixed Nash equilibrium.