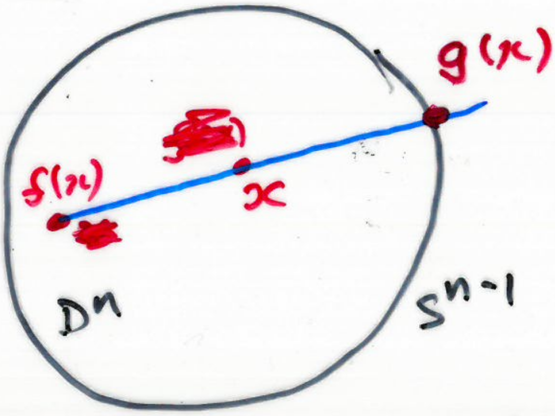
$D^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$

<u>Brouwer's Theorem</u>

Any continuous map $f : D^n \to D^n$ has at least one fixed point $f(x) = x \in D^n$.

<u>Proof</u> suppose f has no fixed point. Then we can define a map

$$g : D^n \longrightarrow S^{n-1}, \quad x \longmapsto g(x)$$



where $g(x)$ is the point in S^{n-1} where the ray from $f(x)$ through x intersects S^{n-1}.

Let $h: S^{n-1} \to D^n$, $x \mapsto x$ be the inclusion.

Note that $gh: S^{n-1} \to S^{n-1}$ is the identity. Thus $gh \simeq 1_{S^{n-1}}$.

Now $hg: D^n \to D^n$ is homotopic to the identity 1_{D^n} via the homotopy

$$H(x,t) = x + t(g(x) - x)$$

for $0 \leq t \leq 1$.

Therefore D^n is homotopy equivalent ~~to~~ to the boundary S^{n-1}.

Therefore, by our major theorem,

$$\chi(D^n) \simeq \chi(S^{n-1}).$$

However, we've seen that

$\chi(D^n) = \chi(*) = 1$, and

$\chi(S^{n-1}) = 0$ or 2.

This contradiction proves the theorem. $\boxtimes$

<u>(Frobenius-) Perron Theorem</u>

Let A be a real $n \times n$ matrix with entries $a_{ij} > 0$ for all i,j. Then A has a positive eigenvalue; Moreover, there is a corresponding eigenvector $v = (x_1, \dots, x_n)$ with each $x_i \geq 0$.

Example $A = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 10$$

$$= \lambda^2 - 5\lambda - 6 \qquad = (\lambda+1)(\lambda-6)$$

So $\lambda = 6$ is a positive eigenvalue, with eigenvector

$v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Proof of theorem

Define $\sigma : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} x_i$

$$\Delta^{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \begin{array}{l} x_i \geq 0 \\ \sum_{i=1}^{n} x_i = 1 \end{array} \right\}$$

Define

$$g : \Delta^{n-1} \to \Delta^{n-1}, \quad x \longmapsto \frac{1}{\sigma(Ax)} Ax$$

Now g is continuous, and Δ^{n-1} is (homeomorphic to) D^{n-1}.

Brouwer's Theorem says that g has a fixed point $x \in \Delta^{n-1}$,

$$x = g(x) = \frac{1}{\sigma(Ax)} Ax.$$

So

$$Ax = \sigma(Ax) . x$$

Therefore x is an eigenvector with eigenvalue $\sigma(Ax) > 0$.