$S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

Each map $f : S^1 \to S^1$ determines a winding number $w(f) \in \mathbb{Z}$.

For a fixed integer n the map

$$g_n : S^1 \to S^1, \quad z \longmapsto z^n$$

has winding number $w(g_n) = n$.

So each integer arises as the winding number of some map $S^1 \to S^1$.

Proposition 2 from last time (!) yields:

if $f : S^1 \to S^1$, $g : S^1 \to S^1$ are homotopic $f \simeq g$ then they have the same winding numbers

$w(f) = w(g)$.

The above gives us

a surjective function

$$w : [S^1, S^1] \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}$$

To show that w is injective we just need to check that if $f : S^1 \to S^1$ with $n = w(f)$ then $f \simeq g_n$ (when $g_n : S^1 \to S^1$, $z \mapsto z^n$). ($n = w(f) = w(f')$ implies $f \simeq g_n \simeq f'$, and $f \simeq f'$.) To see this, let

$$g_n : S^1 \to S^1, \quad e^{i\theta} \longmapsto e^{in\theta}$$

$$f : S^1 \to S^1, \quad e^{i\theta} \longmapsto e^{iF(\theta)}$$

$$H(e^{i\theta}, t) = e^{i\{(1-t)F(\theta) + tn\theta\}}.$$

Application of $[S^1, S^1] \cong \mathbb{Z}$

Fundamental Theorem of Algebra

Any polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$ and of degree $n > 0$ has at least one zero in $\mathbb{C}$.

Proof Since $a_n \neq 0$ a scalar multiple of the polynomial has the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Let's suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

For $\lambda \geq 0$ define the map

$$f_\lambda : S^1 \longrightarrow S^1$$

by

$$f_\lambda(z) = \frac{p(\lambda z)}{|p(\lambda z)|}$$

Any two maps f_λ, $f_{\lambda'}$ are homotopic via the homotopy

$$H_t(z) = \frac{p(((1-t)\lambda + t\lambda')z)}{|p(((1-t)\lambda + t\lambda')z)|}$$

Note that $f_0(z)$ is a constant function and thus has winding number 0.

Exercise: For large λ we have that $f_\lambda(z)$ is homotopic to $g_n : S^1 \to S^1$, $z \mapsto z^n$.

But g_n has $w(g_n) = n$.

And $g_n \simeq f_\lambda \simeq f_0$, and

$w(g_n) = w(f_0) = 0$.

Contradiction.

$\boxtimes$