"Homotopy" is a key notion used in the proof of the topological invariance of the Euler characteristic of a space.

<u>Defn</u> Two maps $f : X \to Y$ and $g : X \to Y$ are <u>homotopic</u> if there exists a continuous map

$$H : X \times [0,1] \to Y, \quad (x,t) \longmapsto H_t(x)$$

such that

$$H_0(x) = f(x), \quad H_1(x) = g(x)$$

for all $x \in X$.

We can think of $H_t(x)$ as a family of maps

$$H_t(x) : X \to Y, \quad x \longmapsto H_t(x).$$

we often refer to H as a <u>homotopy</u>, and write

$$f \simeq g.$$

To fully understand homotopy we need to be clear about the topology on $X \times [0,1]$.

Let $\mathcal{B}$ be the collection of all sets

$$U \times J = \{(u,j) : u \in U, j \in J\}$$

where U is any open subset of X, J is any open subset of $[0,1]$. A set is <u>open</u> in $X \times [0,1]$ if it is a union of sets in $\mathcal{B}$.

Intuitively

$H : X \times [0,1] \longrightarrow Y$

is continuous if a small change in x and t yields only a small change in $H(x,t) = H_t(x)$.

<u>Example</u> Let $Y \subseteq \mathbb{E}^2$ be a convex set. Let X be any topological space.

<u>Any</u> two maps $f : X \to Y$, $g : X \to Y$ are homotopic.

To see this define:

$$H : X \times [0,1] \to Y, \quad (x,t) \longmapsto f(x) + t\,(g(x) - f(x))$$

$$= (1-t) f(x) + t\, g(x)$$

we have $H(x,t) \in Y$ since Y is convex. Also

$$H(x,0) = f(x)$$
$$H(x,1) = g(x).$$

So $f \simeq g$, since H is continuous!

<u>Proposition</u> For fixed spaces X, Y homotopy is an equivalence relation of the collection of maps from X to Y.

<u>Proof</u> For any $f : X \to Y$ we have $f \simeq f$ (reflexive) thanks to the homotopy

$$H_t(x) = f(x).$$

For any $f: X \to Y$, $g: X \to Y$ if $f \simeq g$ then there is a homotopy $H_t(x)$ with $H_0(x) = f(x)$ and $H_1(x) = g(x)$. To see that $g \simeq f$ we define

$$H_t'(x) = H_{1-t}(x).$$

For transitivity, let $f, g, h: X \to Y$ be such that $f \simeq g$ and $g \simeq h$. So we have homotopies

$$H_t(x), \quad H_0(x) = f(x), \; H_1(x) = g(x)$$

$$H_t'(x), \quad H_0'(x) = g(x), \; H_1'(x) = h(x).$$

To see that $f \simeq h$ define

$$H_t''(x) = \begin{cases} H_{2t}(x), & 0 \le t \le \frac{1}{2} \\ H_{2t-1}'(x), & \frac{1}{2} \le t \le 1. \end{cases}$$

$\square$

Let $[f]$ denote the homotopy equivalence class of a map $f: X \to Y$. Let $[X, Y]$ denote the collection of homotopy equivalence classes of maps $X \to Y$.

$[S^3, S^2]$ = bijective with $\mathbb{Z}$

$[S^{14}, S^7]$ = has size 120.