Aims:

- Give a precise definition of the Euler characteristic of a space X
- Give a flavour of the ingredients in the proof that homeomorphic spaces have equal Euler characteristic.
- Give an application of Euler characteristics to Economics.

Simplicial Complexes

Let $v_0, v_1, \ldots, v_k$ be vectors in $\mathbb{E}^n$. These vectors are said to be in general position if the vectors

$$v_1 - v_0,\ v_2 - v_0,\ \ldots,\ v_k - v_0$$

are linear independent.

Example $v_0 = (1,0,0)$, $v_1 = (0,1,0)$, $v_2 = (0,0,1)$

Then

$$v_1 - v_0 = (-1,1,0)$$

$$v_2 - v_0 = (-1,0,1)$$

are linearly independent. In other words

$$\lambda_1 (v_1 - v_0) + \lambda_2 (v_2 - v_0) = (0,0,0)$$

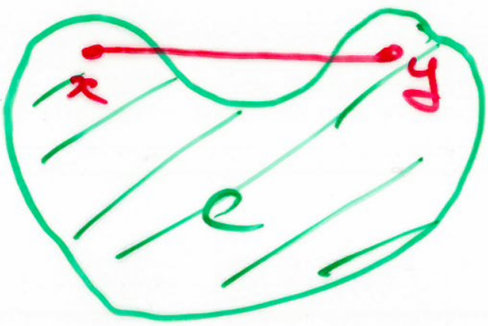
with $\lambda_1, \lambda_2 \in \mathbb{R}$, implies

$$\lambda_1 (-1,1,0) + \lambda_2 (-1,0,1) = (0,0,0)$$

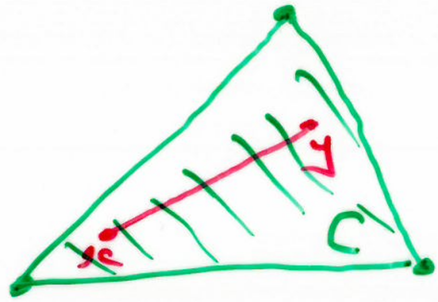
and $\lambda_1 = \lambda_2 = 0$.

So v_0, v_1, v_2 are in general position.

Recall: A set $C \subseteq \mathbb{E}^n$ is said to be <u>convex</u> if, for any points $x, y \in C$, all points on the line from x to y lie in C.



not convex



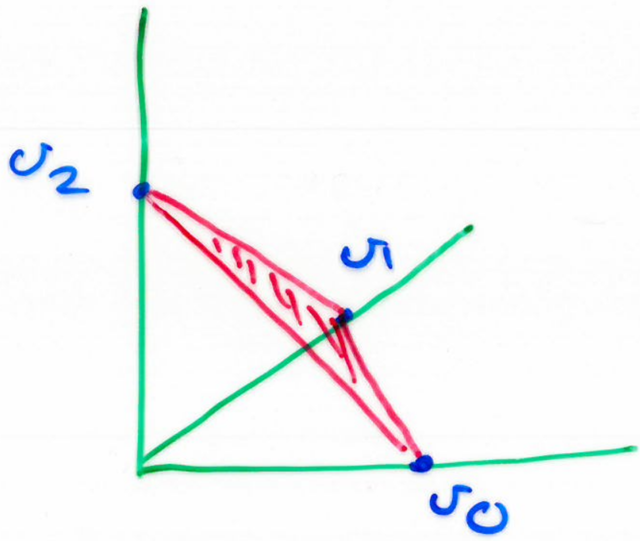
convex

Suppose $v_0, v_1, \dots, v_k \in \mathbb{E}^n$ are in general position. Let

$$C = \mathrm{Convex}(v_0, v_1, \dots, v_n)$$

denote the smallest convex set in $\mathbb{E}^n$ that contains $v_0, v_1, \dots, v_k$.

Example $v_0 = (1,0,0)$, $v_1 = (0,1,0)$, $v_2 = (0,0,1)$



In general, $C = \text{Convex}(v_0, v_1, \dots, v_k)$ consists of all points of the form

$$x = \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_k v_k$$

with $\lambda_i \in \mathbb{R}$, $\lambda_i \geq 0$ and

$$\lambda_0 + \lambda_1 + \dots + \lambda_n = 1.$$

Defn Let $v_0, v_1, \dots, v_k \in \mathbb{E}^n$ be in general position. We call

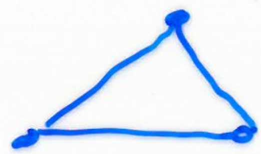
$$C = \text{Convex}(v_0, \dots, v_k)$$

a simplex of dimension k, or k-simplex

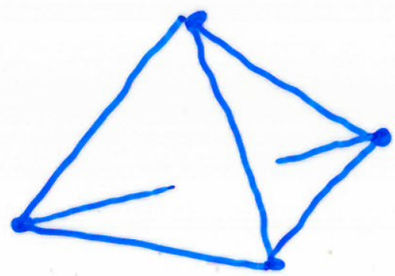
0-simplex = point

1-simplex = line segment

2-simplex = solid triangle



3-simplex = tetrahedron



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simplexes have "faces".

if A and B are simplexes, and if the vertices of A form a subset of the vertices of B, then we say that A is a *face* of B.

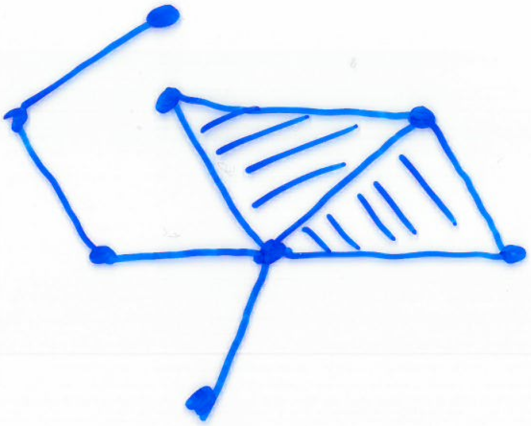
Example A 3-simplex has:

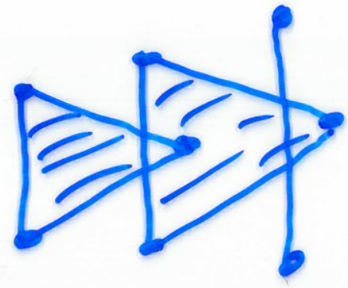
- four faces of dimension 2
- six faces of dimension 1
- four faces of dimension 0

Definition A finite collection of simplexes L in $\mathbb{E}^n$ is called a **simplicial complex** if:

i) whenever a simplex lies in the collection, then so too does all of its faces

ii) whenever two simplexes of the collection intersect, they do so in a common face.





simplicial complex

A simplicial complex is a subset of $\mathbb{E}^n$ and is thus a subspace with the subspace topology.

We K, L .... for simplicial complexes. We write $|K|$, $|L|$ for the corresponding topological subspaces of $\mathbb{E}^n$.

Defn A **triangulation** of a topological space X consists of a simplicial complex K and homeomorphism $h : |K| \longrightarrow X$.