<u>Proposition</u> A set $A \subseteq X$ in a space X is closed if and only if A contains all its accumulation points.

<u>Proof</u> Suppose A is closed.

Then $X \setminus A$ is open.

Any point $x \in X \setminus A$ lies in the open set $X \setminus A$. So no point $x \in X \setminus A$ is an accumulation point. So A contains all its accumulation points.

Conversely, suppose A contains all its accumulation points. Let $x \in X \setminus A$ we can find an open set such that $x \in U_x$ $U_x \subseteq X \setminus A$ because x is <u>not</u> an accumulation point. So

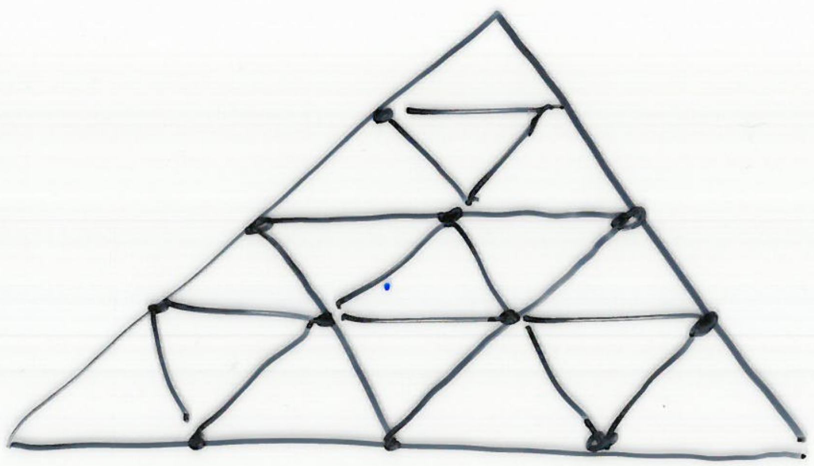
$$X \setminus A = \bigcup_{x \in X\setminus A} U_x .$$

Each U_x is open so the union $X \setminus A$ is open. So A is closed. $\boxed{\text{LLA}}$

Let's explain why the continuous map $f : [0,1] \longrightarrow \Delta$ is surjective.

Recall: $f(x) = \lim_{n \to \infty} f_n(x)$

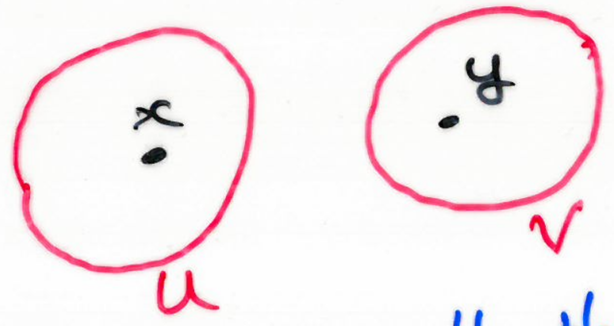
where $f_n : [0,1] \to \Delta$ is defined by dividing Δ into small equilateral triangles of side $\frac{1}{2^n}$.



Any point in Δ lies in infinitely many triangles of side $\frac{1}{2^n}$, $n = 1, 2, 3, 4, \cdots$, So any point in Δ is an accumulation point for $f([0,1])$.

We know that $[0,1]$ is compact. since f is continuous, we know that $f([0,1])$ is compact. we just need to prove that $f([0,1])$ is closed.

<u>Defn</u> A topological space X is said to be <u>Hausdorff</u> if for any distinct points $x, y \in X$ there exist open sets U, V with:
$x \in U$, $y \in V$, $U \cap V = \phi$.



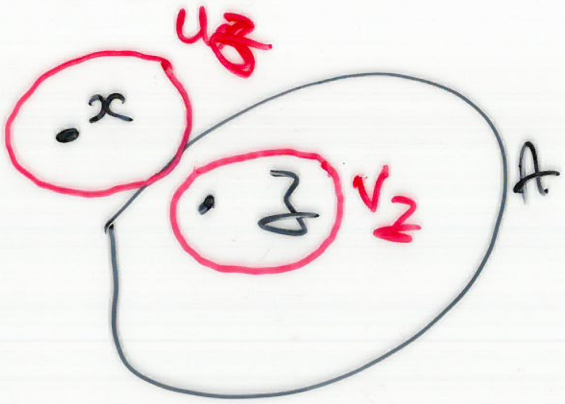
<u>Example</u> $\mathbb{R}$ with the usual topology is Hausdorff. So too is $\mathbb{R}^n$ for $n \geq 1$.

<u>Example</u> If we give $\mathbb{R}$ the cofinite topology: a set is open iff its complement is finite.
This topology on $\mathbb{R}$ is <u>not</u> Hausdorff.

<u>Proposition</u> A compact subset of a Hausdorff topological space is closed.

<u>Proof</u> Let X be a Hausdorff space. Let A be a compact subset. Let $x \in X \setminus A$. We just need to show that x is not an accumulation point.

Let $z \in A$. We can find open sets U_z, V_z with $z \in V_z$, $x \in U_z$, $V_z \cap U_z = \emptyset$.



we have a collection of open sets

$$\{ V_z \}_{z \in A} .$$

The union of this collection contains A. But, A is compact. So, we can find a finite set of points $z_1, z_2, \cdots, z_k \in A$ with

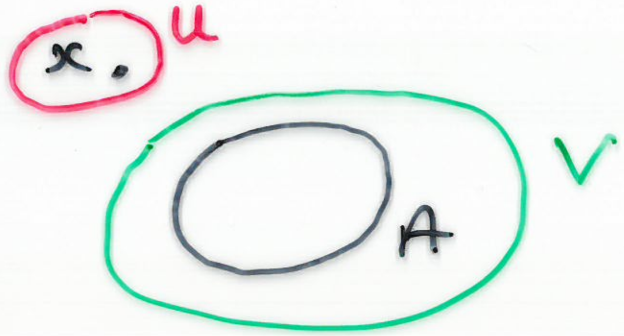
$$A \subseteq V_{z_1} \cup V_{z_2} \cup \cdots \cup V_{z_k} .$$

Let $V = V_{z_1} \cup V_{z_2} \cup \cdots \cup V_{z_k}$

Now V is disjoint from the finite intersection

$$U = U_{z_1} \cap U_{z_2} \cap \cdots \cap U_{z_k} .$$

But U is open since it is a finite intersection of open sets



Hence $U \cap A = \phi$ and x is <u>not</u> an accumulation point. $\square$