Theorem $[0,1]$ is a compact subspace of $\mathbb{R}$.

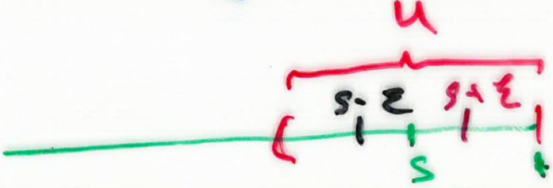
Proof Let $\mathcal{F}$ be an open cover of $[0,1]$.

Define
$$X = \{ x \in [0,1] : [0,x] \text{ is contained in a finite subcover of } \mathcal{F} \}.$$

- X is non-empty since $0 \in X$.
- If $0 \le y \le x$ and if $x \in X$ then $y \in X$.
- X has a least upper bound s since X is bounded above by 1. (Completeness Axiom for the real numbers.)

We need to prove $s = 1$ and $s \in X$.

Choose $U \in \mathcal{F}$ with $s \in U$.



~~Therefore~~ choose $\varepsilon > 0$ with $s - \varepsilon \in U$.

Then there is a finite subcover

$\mathcal{F}'$ containing $[0, s-\varepsilon]$.

So $\mathcal{F}' \cup \{U\}$ is a finite subcover containing $[0, s]$. So $s \in X$.

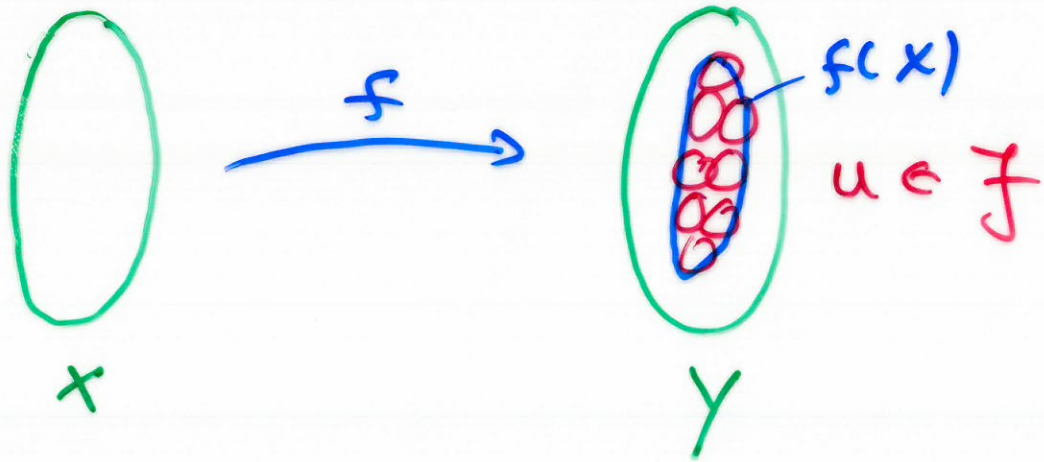
If $s < 1$ then can suppose $s + \varepsilon \in U$ for sufficiently small $\varepsilon > 0$. So $s + \varepsilon \in U$. This contradicts the fact that s is the l.u.b. of X. Hence $s = 1$.

$\blacksquare$

Compactness is a topological property because:

<u>Proposition</u> Suppose $f : X \to Y$ is a continuous map. If X is compact then so too is the image $f(X)$.

Proof Let $\mathcal{F}$ be an open cover of the image of $f(X)$.



Then $\{f^{-1}(U)\}_{U \in \mathcal{F}}$ is an open cover of X. If X is compact then there is a finite subcover $\{f^{-1}(U)\}_{U \in \mathcal{F}'}$ of X. But then $\{U\}_{U \in \mathcal{F}'} = \mathcal{F}'$ is a finite cover of $f(X)$. Hence $f(X)$ is compact. $\blacksquare$

<u>Definition</u> Let X be a topological space. A subset $A \subseteq X$ is <u>closed</u> if its complement $X \setminus A$ is an open subset of X.

<u>Example</u> $[0,1] \subseteq \mathbb{R}$ is closed since $\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$ is open.

<u>Example</u> $(0,1] \subseteq \mathbb{R}$ is neither open nor closed.

Defn Let A be a subset of a topological space X. A point $p \in X$ is an accumulation point of A if every open set $U \in X$ that contains p also contains some point in $A \setminus \{p\}$.

Example Let $X = \mathbb{R}$, $A = \left\{ \frac{1}{n} \right\}_{n=1,2,3,\dots}$. In this example 0 is (the only) accumulation point of A.

Example Let $X = \mathbb{R}$, $A = [0, 1)$. Then every point in A is an accumulation point. So too is 1.

Proposition A set $A \subseteq X$ is closed if and only if A contains all its accumulation points.