Let's aim for a proof that $[0,1]$ is <u>not</u> homeomorphic to Δ.

<u>Proposition</u> Let $f : X \to Y$ be a homeomorphism. Then X is connected if and only if Y is connected.

<u>Proof</u> Suppose Y is <u>not</u> connected. Then there exist open subsets $U, V \subset Y$ such that U, V are both non-empty, and $Y = U \cup V$ and $U \cap V = \emptyset$. Since f is continuous we see that

$$f^{-1}(U) = \{x \in X : f(x) \in U\}$$

is open in X. So too is $f^{-1}(V)$ open in X.

Since f is a homeomorphism, there is a continuous map $g: Y \to X$ with $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in X$, $y \in Y$.

Since U, V are non-empty, then $g(U) = f^{-1}(U)$, $g(V) = f^{-1}(V)$ are non-empty. Also

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset$$

clearly!, and $X = f^{-1}(U) \cup f^{-1}(V)$.

So X is <u>not</u> connected.

By swapping the roles of f and g we get that if X is <u>not</u> connected then Y is not connected.

$\boxtimes$

Theorem $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

sketch proof

Suppose there were a homeomorphism

$$f : \mathbb{R} \longrightarrow \mathbb{R}^2 .$$

Consider

$$X = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$$

$$Y = \mathbb{R}^2 \setminus \{ f(0) \} .$$

Excercise: The function

$$f' : X \to Y, \quad x \mapsto f(x)$$

is a homeomorphism.

But X is <u>not</u> connected while Y is connected. This contradicts the above proposition.

Compactness

Let X be a topological space. Let $\mathcal{F}$ be a family of open subsets of X whose union equals X. Then we say that $\mathcal{F}$ is an open cover of X.

<u>Example</u> $X = \mathbb{R}$. Let

$$\mathcal{F} = \{ (n-2, n+2) \}_{n \in \mathbb{Z}} .$$

Then $\mathcal{F}$ is an open cover of $X = \mathbb{R}$.

If $\mathcal{F}'$ is a subfamily of $\mathcal{F}$, and if the union of all sets in $\mathcal{F}'$ equals X then we

say that $\mathcal{F}'$ is a <u>subcover</u> of $\mathcal{F}$.

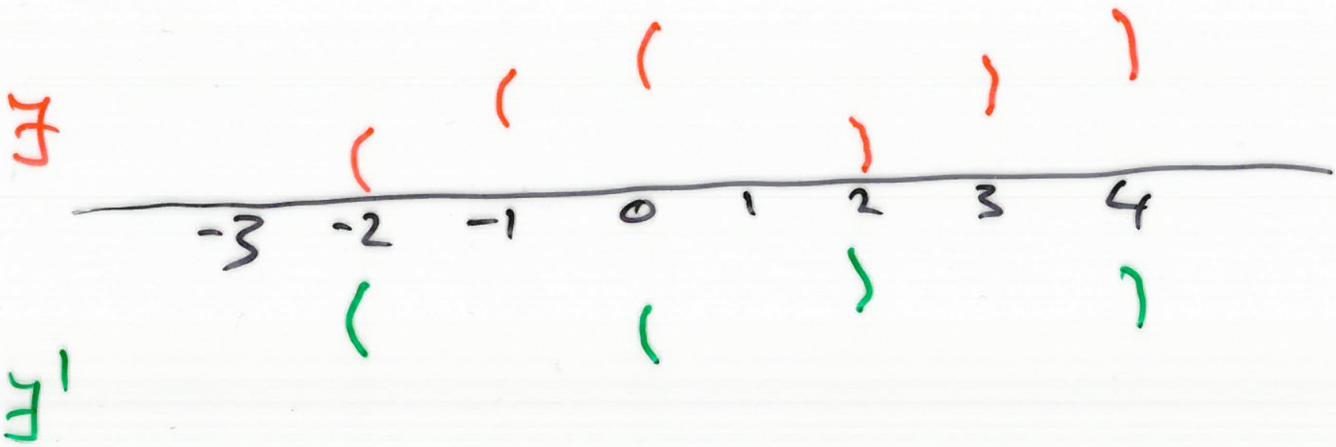
<u>Example</u> (continued)

Let $2\mathbb{Z}$ be the even integers.

Then

$$\mathcal{F}' = \{(n-2, n+2)\}_{n \in 2\mathbb{Z}}$$

is a subcover of the above $\mathcal{F}$.



An open cover $\mathcal{F}$ of X is said to be <u>finite</u> if $\mathcal{F}$ consists of only finitely many open sets.

Definition A topological space X is compact if every open cover has a finite subcover.

Example The space $X = \mathbb{R}$ is <u>not</u> compact. To see this, just note that the open cover

$$\mathcal{F} = \{(n-2, n+2)\}_{n \in \mathbb{Z}}$$

has no finite subcover.

Theorem The closed interval $[a,b]$ in $\mathbb{R}$ <u>is</u> compact.

Proof (Maybe next time?)

Theorem Suppose that X is homeomorphic to Y. Then X is compact if and only if Y is compact.

proof (next time)