<u>Example</u> The unit circle

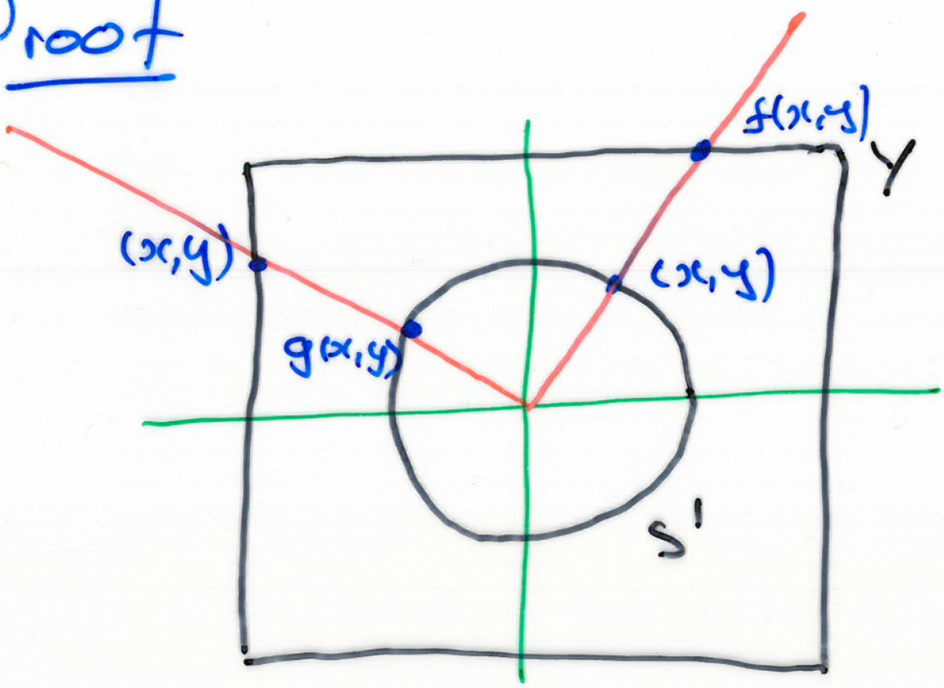
$$S^1 = \{(x,y) \in \mathbb{E}^2 : x^2+y^2=1\}$$

is homeomorphic to the square

Y of side 2,

$$Y = \{(x,y) \in \mathbb{E}^2 : -2 \le x,y \le 2 \text{ and either } x \in \{-2,2\} \text{ or } y \in \{-2,2\}\}.$$

<u>Proof</u>



Consider $f : S^1 \to Y$, $(x,y) \mapsto f(x,y)$

where $f(x,y)$ is the intersection

of the ray from the origin through (x,y) and the space Y.
This function is continuous.

Consider $g : Y \to S^1$, $(x,y) \mapsto g(x,y)$ where $g(x,y)$ is the intersection of the ray through the origin and (x,y) and the space S^1.

Note: g is continuous, and

$$g(f(x,y)) = (x,y)$$

$$f(g(x,y)) = (x,y).$$

Hence the square and circle are homeomorphic. $\blacksquare$

<u>Example</u> A doughnut is homeomorphic to a coffee cup.

<u>Proposition</u> If $f : X \to Y$ and $h : Y \to Z$ are continuous functions then their composite $h \circ f : X \to Z$, $x \mapsto h(f(x))$ is continuous.

<u>Proof</u> Let $U \subset Z$ be open in Z. Then $h^{-1}(U) \subset Y$ is open in Y since h is continuous. So $f^{-1}(h^{-1}(U)) \subset X$ is open in X by continuity of f.

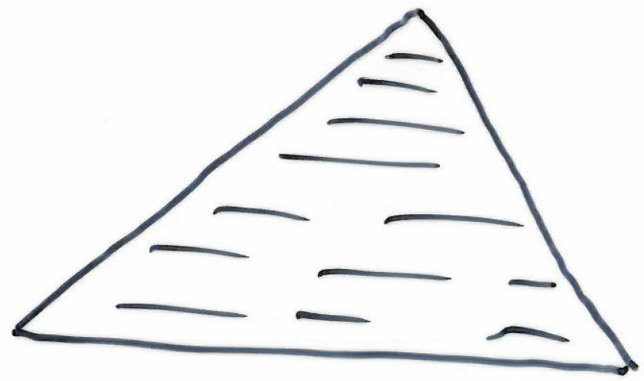
But $f^{-1}(h^{-1}(U)) = (h \circ f)^{-1} U$.

So the pre-image under $h \circ f$ of an open set $U \subset Z$ is an open set $(h \circ f)^{-1} U \subset X$.

$\square$

A Space-filling curve (Peano curve)

Let Δ be an equilateral triangular region in $\mathbb{E}^2$ of side 1.

Δ =  $\subseteq \mathbb{E}^2$.

Theorem (Peano) There exists a surjective continuous function

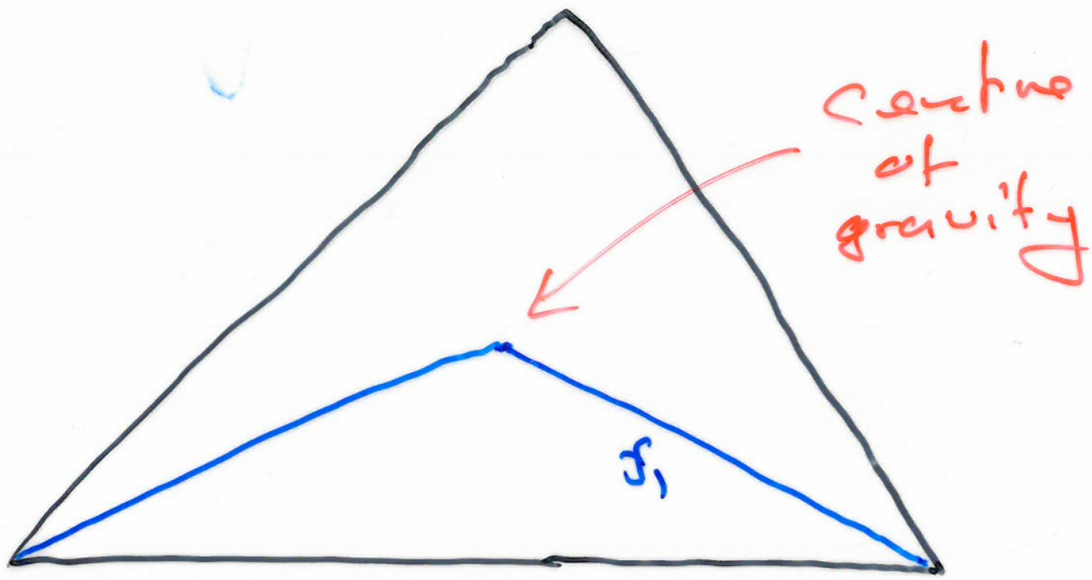
$$f : [0,1] \longrightarrow \Delta$$

Proof We first construct a sequence of continuous

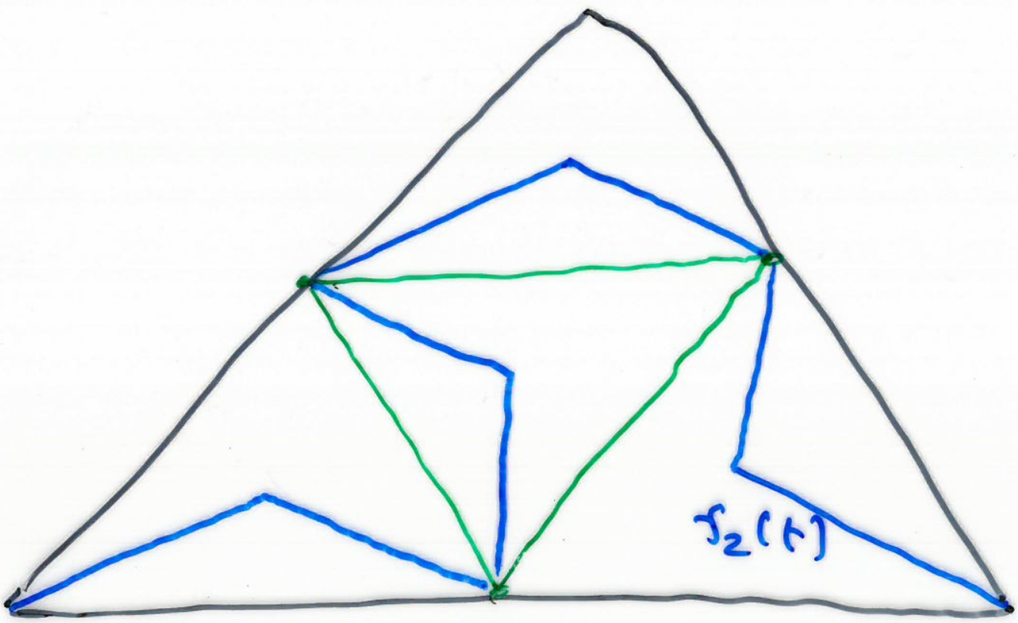
functions $f_1 : [0,1] \longrightarrow \Delta$,

$f_2 : [0,1] \longrightarrow \Delta$, ...

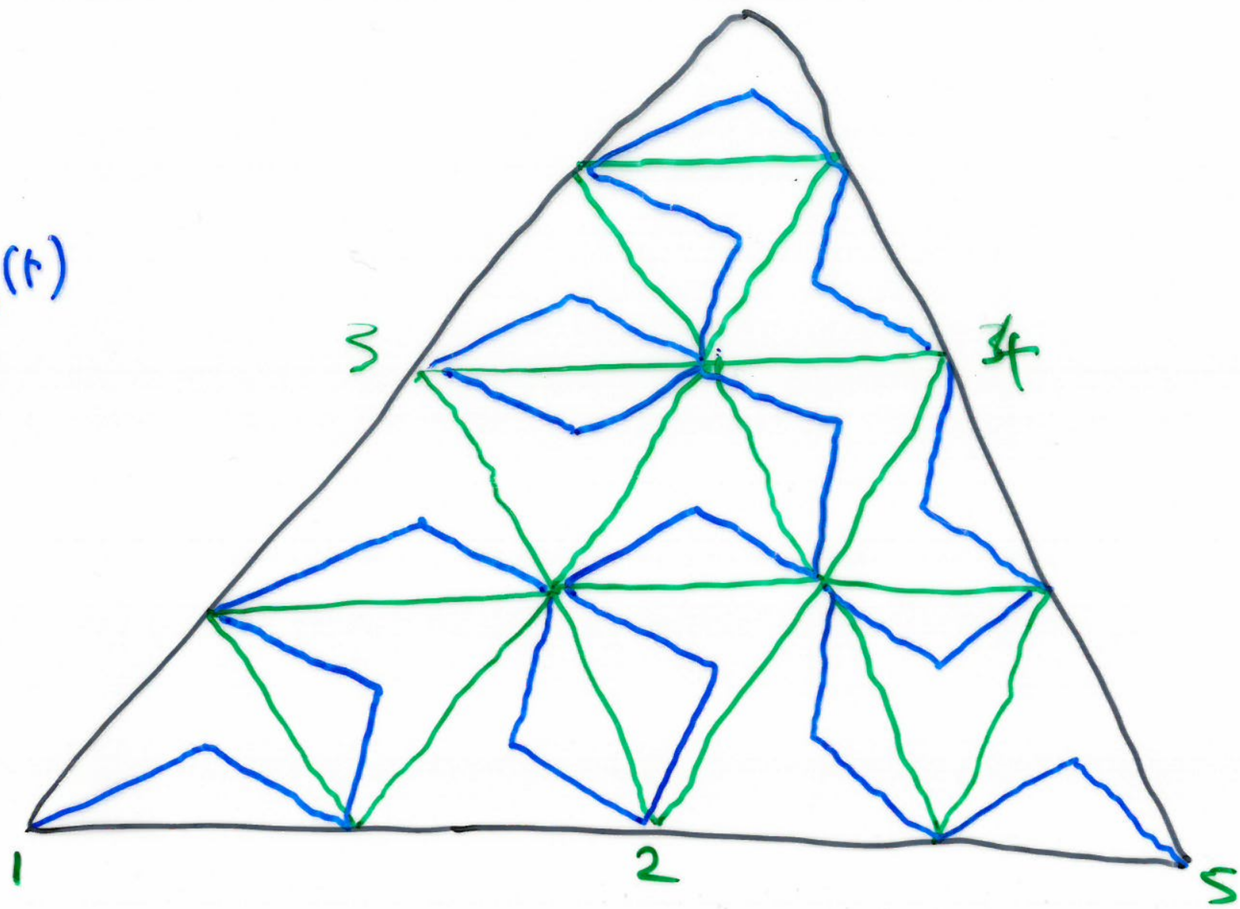
$f_1(t)$



$f_2(t)$



$f_3(t)$



For $f_n : [0,1] \longrightarrow \Delta$ we subdivide Δ into 4^{n-1} green triangles, and the image of $f_n : [0,1] \longrightarrow \Delta$ inside each of the 4^{n-1} triangles looks just like f_1.

To complete the proof we:

1) Define f to be the limit of $f_1, f_2, f_3, \ldots$. (we need to check that limit exists.)

2) Need to prove that f is onto. (we'll need to use the notion of "compactness"!)