<u>Defn</u> Let X be a set with topology T. Let $Y \subseteq X$ be a subset. In the <u>subspace topology</u> on Y a subset $U \subseteq Y$ is open if

$$U = Y \cap A$$

where A is an open subset of X. With this topology we call Y a <u>subspace</u> of X.

<u>Example</u> For the real line $\mathbb{E}^1$ we have the subset $\mathbb{Q} \subset \mathbb{E}^1$. With the subspace topology, $\mathbb{Q}$ is <u>not</u> connected.

To see this consider

$$A = \{ x \in \mathbb{R} : x > \sqrt{2} \} = (\sqrt{2}, \infty)$$

$$B = \{ x \in \mathbb{R} : x < \sqrt{2} \} = (-\infty, \sqrt{2})$$

Then

$$\mathbb{Q} = (A \cap \mathbb{Q}) \cup (B \cap \mathbb{Q})$$

and

$$(A \cap \mathbb{Q}) \cap (B \cap \mathbb{Q}) = \emptyset$$

and $A \cap \mathbb{Q}$, $B \cap \mathbb{Q}$ are open in $\mathbb{Q}$. Hence $\mathbb{Q}$ is <u>not</u> connected.

<u>Defn</u> Let X be a topological space. Let $Y \subseteq X$ be a subset. We say that the subset Y is <u>connected</u> if and only if it is connected as a subspace.

Defn A connected component of a topological space X is a connected subspace $Y \subseteq X$ such that there is no connected subspace $W \subseteq X$ with $Y \subsetneq W$.

Example

Let $X = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 \neq 1\}$

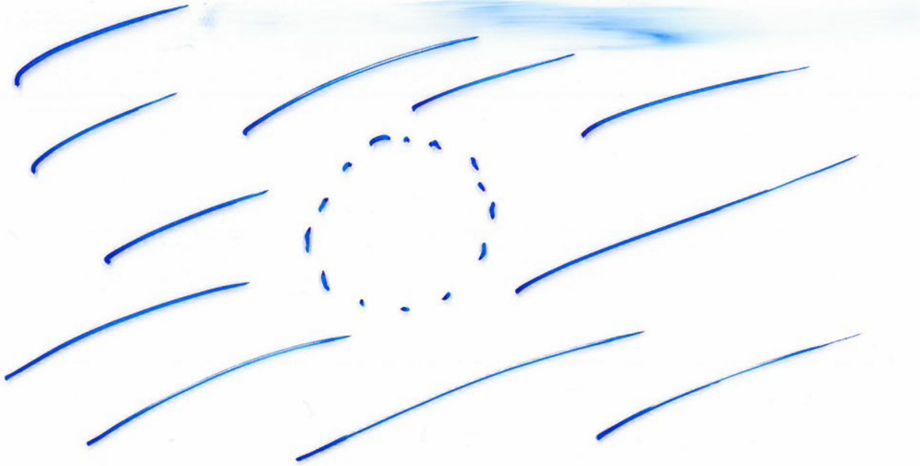
There are two connected components of X, namely

$$Y = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 > 1\}$$

and

$$Z = \{(x,y) \in \mathbb{E}^2 : x^2+y^2 < 1\}.$$

Y



Z

Example

Let $X = \mathbb{Q} \subset \mathbb{E}^1$. Then the connected components of $\mathbb{Q}$ are the sets $\{x\}$ for each $x \in \mathbb{Q}$.

Introduction to Topological Data Analysis

	H	M	R	C	W
H	0	11	10	14	22
M	11	0	3	13	21
R	10	3	0	12	20
C	14	13	12	0	16
W	22	21	20	16	0

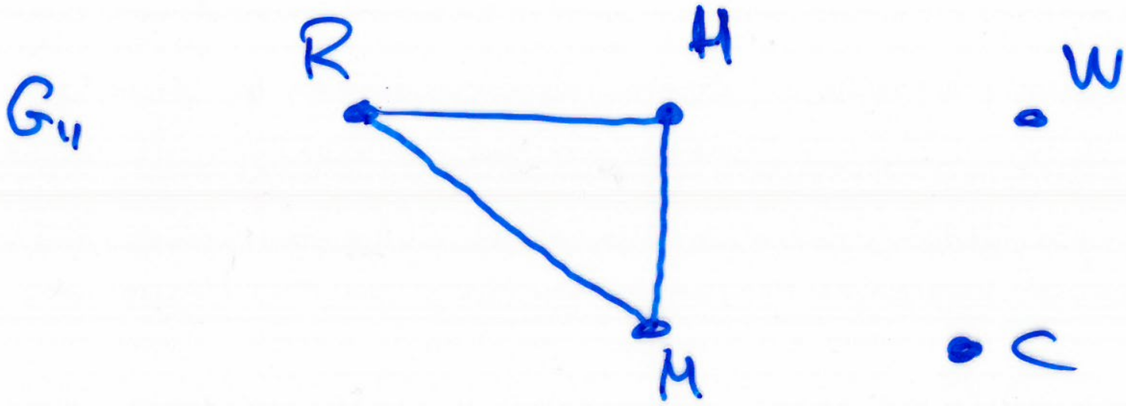
- Human, Mouse, Rat, Cat, Whale
- Harrods, M&S, Restaurant Group, Coca-Cola HBC, Whitbread.

$$\mathrm{dist}(H, R) = \mathrm{dist}(R, H)$$

Choose some threshold $\varepsilon > 0$ and consider the graph G_ε with vertices H, M, R, C, W and with an edge

if $\operatorname{dist}(X, Y) \leq \varepsilon$.

For $\varepsilon = 11$ we have



We regard this graph as a subspace of $\mathbb{E}^5$ by identifying

$$H = (1,0,0,0,0) = e_1$$

$$M = (0,1,0,0,0) = e_2$$

$$R = (0,0,1,0,0) = e_3$$

$$C = (0,0,0,1,0) = e_4$$

$$W = (0,0,0,0,1) = e_4$$

The graph G_{11} can be thought of as the subspace of $\mathbb{E}^5$ with points e_1, e_2, e_3, e_4, e_5 and line segments

$$e_1e_3\ ,\ e_1e_2\ ,\ e_2e_3\ .$$

The topological space G_{11} has precisely <u>three</u> connected components.