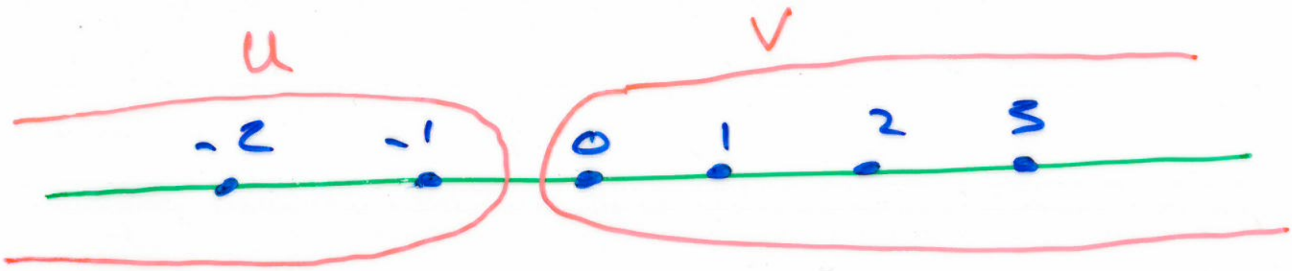
Example (revision)

The set of integers $\mathbb{Z} \subseteq \mathbb{E}^1$ is <u>not</u> a connected subset of the real line $\mathbb{E}^1$,



because we can find two open subsets

$$U = \{ x \in \mathbb{R} : x < -\tfrac{1}{2} \}$$

$$V = \{ x \in \mathbb{R} : x > -\tfrac{1}{2} \}$$

such that

$$U \cup V \supseteq \mathbb{Z},$$

and

$$\mathbb{Z} \cap U \cap V = \emptyset,$$

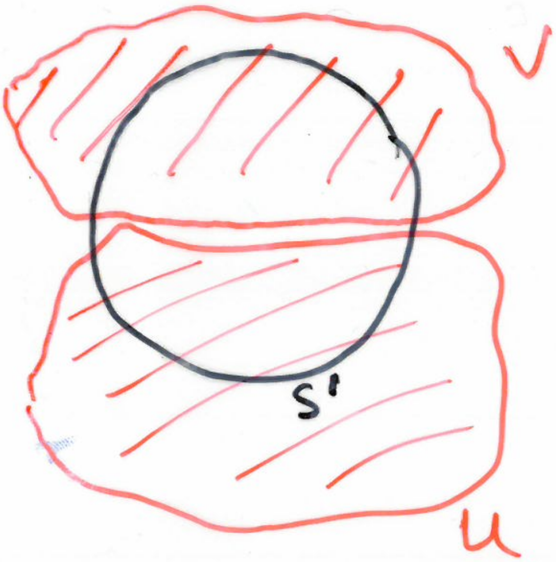
and

$$\mathbb{Z} \cap U \neq \emptyset \neq \mathbb{Z} \cap V.$$

Example (revision)

$$S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2=1 \}$$

<u>is</u> a connected subset of the Euclidean plane $\mathbb{E}^2$.



If U, V are non-empty open subsets of the plane with $S^1 \cap U \cap V = \emptyset$, then it is not possible to have $S^1 \subset U \cup V$.

Definition (Riesz [1909], Hausdorff [1914])

A topological space consists of a set X and a collection T of subsets of X which we deem to be "open". The following axioms must hold.

T1) The union of any collection of open sets is open.

T2) The intersection of any finite collection of open sets is open.

T3) Both $\emptyset$ and X are open.

<u>Definition</u> A topological space X is said to be <u>connected</u> if there does not exist two non-empty open sets U, V in X such that

$$U \cup V = X$$

and

$$U \cap V = \phi$$

<u>Example</u> Let $X = \mathbb{R}^n$

$$= \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$$

Let $\mathcal{T}$ consist of those subsets $U \subseteq \mathbb{R}^n$ such that for any $x \in U$ we can find $\varepsilon > 0$ such that the Euclidean open ball $B^n(x, \varepsilon)$ of radius ε centred at x

lies entirely in U.

$$B^n(x, \varepsilon) \subseteq U.$$

<u>Example</u> Let $X = \mathbb{R}^n$. Let T consist of <u>all</u> subsets of X. This is again a topological space. We call this topology T the <u>discrete topology</u>. This space is <u>not</u> connected. For instance

$$U = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$$

$$V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}$$

Then U, V are non-empty, open with $U \cup V = X$, $U \cap V = \emptyset$.

Example Let $X = \mathbb{R}^n$. Let T consist of just two open sets: $\{\phi, X\} = T$. This is a topological space. We call T the **trivial topology**. This topological space is connected.

Example Let $X = \mathbb{Z}$. The **cofinite topology** on X has as open sets those subsets $U \subseteq X$ such that the complement $X \setminus U$ is finite. Also, ϕ is deemed to be open.

This is a topological space. With this topology

$\mathbb{Z}$ is connected.