$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

<u>Jordan Curve Theorem</u>

Let $\alpha : S^1 \longrightarrow \mathbb{R}^2$ be an injective continuous function. Let $J \in \mathbb{R}^2$ be the image of α. Then $\mathbb{R}^2 \setminus J$ has precisely two connected components both of which have frontier J.

Aim for next few lectures:

1) Explain underlined terms

2) Give the explanation using the notion of a "continuous function between topological spaces"

$$f : X \longrightarrow Y$$

3) Give some weird examples that suggest the theorem is not obvious

4) A proof of the theorem is just beyond the scope of this course. But we'll prove an easy special case.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we define the Euclidean norm to be

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

For $x, y \in \mathbb{R}^n$ we define the Euclidean distance

$$d(x,y) = \|x - y\|$$

We write $\mathbb{E}^n$ denote the set $\mathbb{R}^n$ endowed with the Euclidean distance.

Also we define the n-sphere to be

$$S^n = \{ x \in \mathbb{E}^{n+1} : \|x\| = 1 \}$$

S^1: 

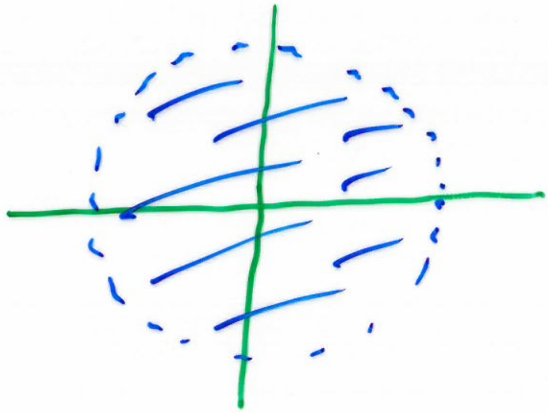
S^2: 

For $x \in \mathbb{E}^n$ and for any real number $\varepsilon > 0$ we define the open ball centred at x of radius ε to be

$$B^n(x, \varepsilon) = \{ y \in \mathbb{E}^n : d(x,y) < \varepsilon \}$$

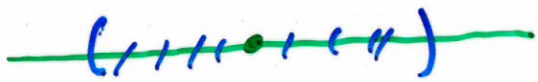
Examples

$B^2((0,0), 1)$



$B^1(0, 1)$

$= (-1, 1)$

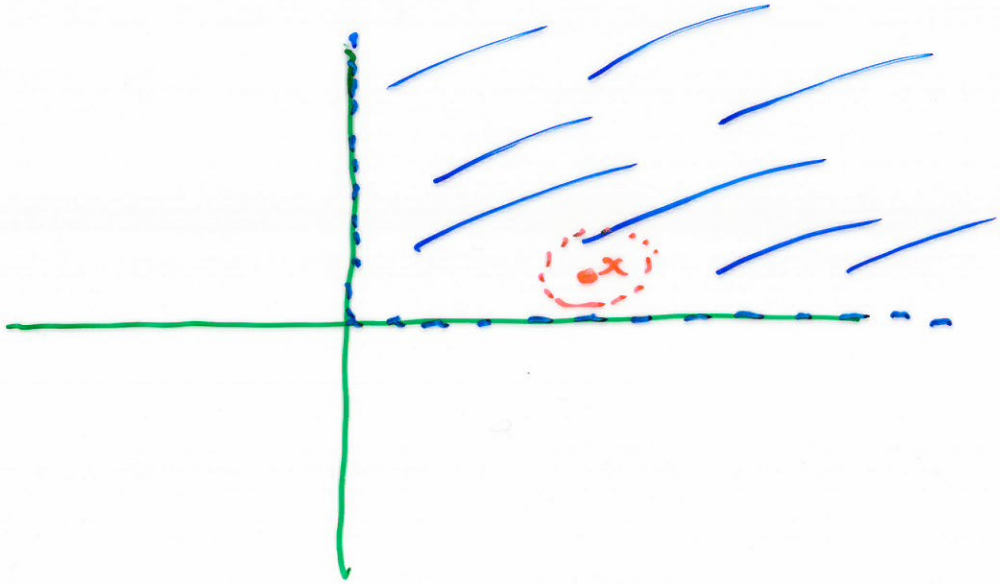


Defn A set $X \subseteq \mathbb{E}^n$ is said to be open if, for any $x \in X$, we can find some $\varepsilon > 0$ (ε can depend on x) such that

$$B^n(x, \varepsilon) \subseteq X.$$

Example

Consider $X = \{(x,y) \in \mathbb{E}^2 : x > 0, y > 0\}$



Then X is an open subset of $\mathbb{E}^2$

Example

Consider

$$X = \{x \in \mathbb{R} : 0 < x < 1\}$$
$$= (0,1)$$

Then $X = (0,1)$ is an open subset of $\mathbb{E}^1$.

Example S^2 is **not** an open subset of $\mathbb{E}^3$.

Three important facts

Fact 1 if $X_1, X_2, \ldots$ is a collection of open sets in $\mathbb{E}^n$ then their union

$$X_1 \cup X_2 \cup \cdots$$

is an open set in $\mathbb{E}^n$.

Fact 2 if $X_1, X_2, \ldots, X_n$ is a finite collection of open sets in $\mathbb{E}^n$ then their intersection

$$X_1 \cap X_2 \cap \cdots \cap X_n$$

is an open set in $\mathbb{E}^n$.

Fact 3 The empty set $\emptyset$ and the set $X = \mathbb{E}^n$ are open in $\mathbb{E}^n$.

Remark Fact 2 does <u>not</u> extend to infinite collections, consider

$$X_i = (0, 1 + \tfrac{1}{i}) \subseteq \mathbb{E}^1 \quad , i = 1, 2, 3, \dots .$$

Then

$$\bigcap_{i \geq 1}^{\infty} X_i = (0, 1]$$

is <u>not</u> open.

Defn A subset $X \subseteq \mathbb{E}^n$ is **connected** if there do not exist open subsets U, V in $\mathbb{E}^n$ such that

$$U \cup V \supseteq X$$

and

$$U \cap X \neq \emptyset \neq V \cap X$$

and

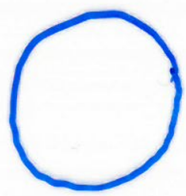
$$U \cap V \cap X = \emptyset .$$

Example

Consider $X = \{x \in \mathbb{E}^2 : \|x\| \neq 1\}$.

Then X is **not** connected.

To see this, take



$U = \{x \in \mathbb{E}^2 : \|x\| < 1\}$

$V = \{x \in \mathbb{E}^2 : \|x\| > \}$

Then U, V are open, $U \cup V = X$, $U \cap X \neq \emptyset \neq V \cap X$ and $U \cap V \cap X = \emptyset$.