Game Theory

A game involves

- n players
- a set S_i of strategies for player i.
- a payoff function
  $$v_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$
  for player i, $1 \le i \le n$.

Example

2 players, Mary & John who want to go to either the cinema (C) or soccer match (S) together.

$$S_1 = \{C, S\}$$
$$S_2 = \{C, S\}$$

Payoff:

$v_1(C, C) = 2$ $v_2(C, C) = 1$	$v_1(C, S) = 0$ $v_2(C, S) = 0$
$v_1(S, C) = 0$ $v_2(S, C) = 0$	$v_1(S, S) = 1$ $v_2(S, S) = 2$

Example 2

2 players, each places a coin on the table. Player 1 wants coins to be the same; player 2 wants coins to be different.

$S_1 = \{H, T\}$

$S_2 = \{H, T\}$

payoff:

$v_1(H,H) = 1$ $v_2(H,H) = -1$	$v_1(H,T) = -1$ $v_2(H,T) = 1$
$v_1(T,H) = -1$ $v_2(T,H) = 1$	$v_1(T,T) = 1$ $v_2(T,T) = -1$

In a pure strategy game each player decides, beforehand, on one strategy to play. A pure Nash equilibrium occurs if,

having played the game, no player benefits from unilaterally changing his/her choice of strategy.

Example 1 There are two pure Nash equilibria: both go to the cinema, or both go to soccer match.

Example 2 There is no pure Nash equilibrium in this game.

In a mixed strategy game player i decides on a probability $p_i(s)$ with which to play strategy $s \in S_i$.

So

$$p_i(s) \geq 0 \quad \text{and} \quad \sum_{s \in S_i} p_i(s) = 1.$$

The ith player wants to maximize the expected payoff

$$E(\nu_i(s_1, \dots, s_n)) = \sum_{\substack{s_1 \in S_1 \\ s_2 \in S_2 \\ \vdots \\ s_n \in S_n}} p_1(s_1) p_2(s_2) \cdots p_n(s_n) \nu_i(s_1, \dots, s_n).$$

A mixed Nash equilibrium occurs if, having played the game, no player benefits by unilaterally changing his/her mixed strategy.

Theorem (J. Nash) In any game with finitely many players and finite pure strategy sets S_i, there exists a mixed Nash equilibrium.

Idea of proof

A mixed strategy for player i is a vector $p_i \in \mathbb{R}^{|S_i|}$.

Consider the set

$$C = \{(p_1, \dots, p_n)\} \subseteq \mathbb{R}^{|S_1|+|S_2|+\dots+|S_n|}.$$

Now C is closed, bounded and convex since $p_i(s) \geq 0$,

$$\sum_{s \in S_i} p_i(s) = 1.$$

Given $(p_1, p_2, \dots, p_n) \in C$ define

q_i.

A fixed point of the function

$$f : C \to C, \ (p_1, \dots, p_n) \mapsto (q_1, q_2, \dots, q_n)$$

is a mixed Nash equilibrium.

Such a fixed point exists by

Brouwer's Theorem.

However, to make this into a genuine proof, we need to overcome some "minor" technical issues (such as q_i may not be unique.)