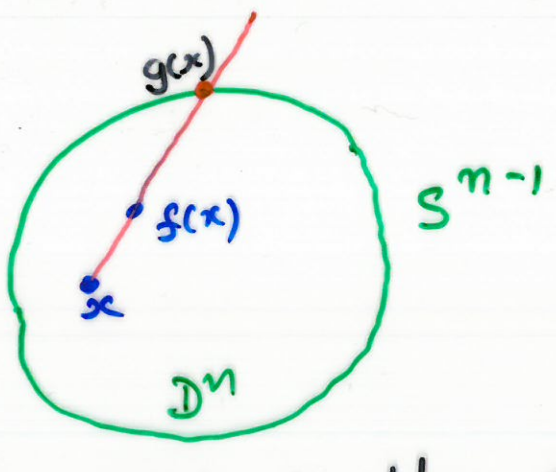
$D^n = \{ x \in E^n : \|x\| \leq 1 \}$

Brouwer's Theorem

Any continuous map $f : D^n \to D^n$ has at least one fixed point $f(x) = x \in D^n$.

Proof suppose a map $f : D^n \to D^n$ has no fixed point. Then we can define a map

$$g : D^n \longrightarrow S^{n-1}, \quad x \longmapsto g(x)$$



where $g(x)$ is the point in S^{n-1} where the ray from x through $f(x)$ intersects S^{n-1}.

Let $h: S^{n-1} \to D^n$, $x \mapsto x$

be the inclusion.

Note that $gh: S^{n-1} \to S^{n-1}$ is

the identity $1_{S^{n-1}}: S^{n-1} \to S^{n-1}$.

Thus $gh \simeq 1_{S^{n-1}}$.

Now $hg: D^n \to D^n$ is homotopic

to the identity 1_{D^n} via

the homotopy

$$H(x,t) = x + t(g(x) - x)$$

for $0 \leq t \leq 1$.

Therefore D^n is homotopy

equivalent to S^{n-1}.

Therefore, by our major theorem,

$$\chi(D^n) = \chi(S^{n-1}).$$

However, we've seen $\chi(D^n) = 1$, $\chi(S^{n-1}) = 0$ or 2.

This contradiction proves the theorem. $\boxed{\text{✓}}$

<u>(Frobenius-)Perron Theorem:</u>

Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ij} > 0$ for each i, j. Then A has a positive eigenvalue; moreover, there is a corresponding eigenvector $v = (x_1, \ldots, x_n)$ with each $x_i \geq 0$.

Example $A = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 10$$

$$= \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6).$$

So $\lambda = 6$ is an eigenvalue.

A corresponding eigenvector is $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Proof of theorem

Define $\sigma : \mathbb{R}^n \to \mathbb{R}, (x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} x_i$.

$$\Delta^{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \begin{array}{l} x_i \geq 0 \\ \sum_{i=1}^{n} x_i = 1 \end{array} \right\}$$

Define

$$g : \Delta^{n-1} \to \Delta^{n-1}, \quad x \longmapsto \frac{1}{\sigma(Ax)} Ax .$$

Now g is continuous, and Δ^{n-1} is (homeomorphic to) D^{n-1}. Brouwer's Theorem implies that g has a fixed point $x \in \Delta^{n-1}$,

$$x = g(x) = \frac{1}{\sigma Ax} Ax$$

Thus $Ax = (\sigma Ax)\, x$.

Therefor x is an eigenvector with eigenvalue $\sigma(Ax)$.

$\blacksquare$