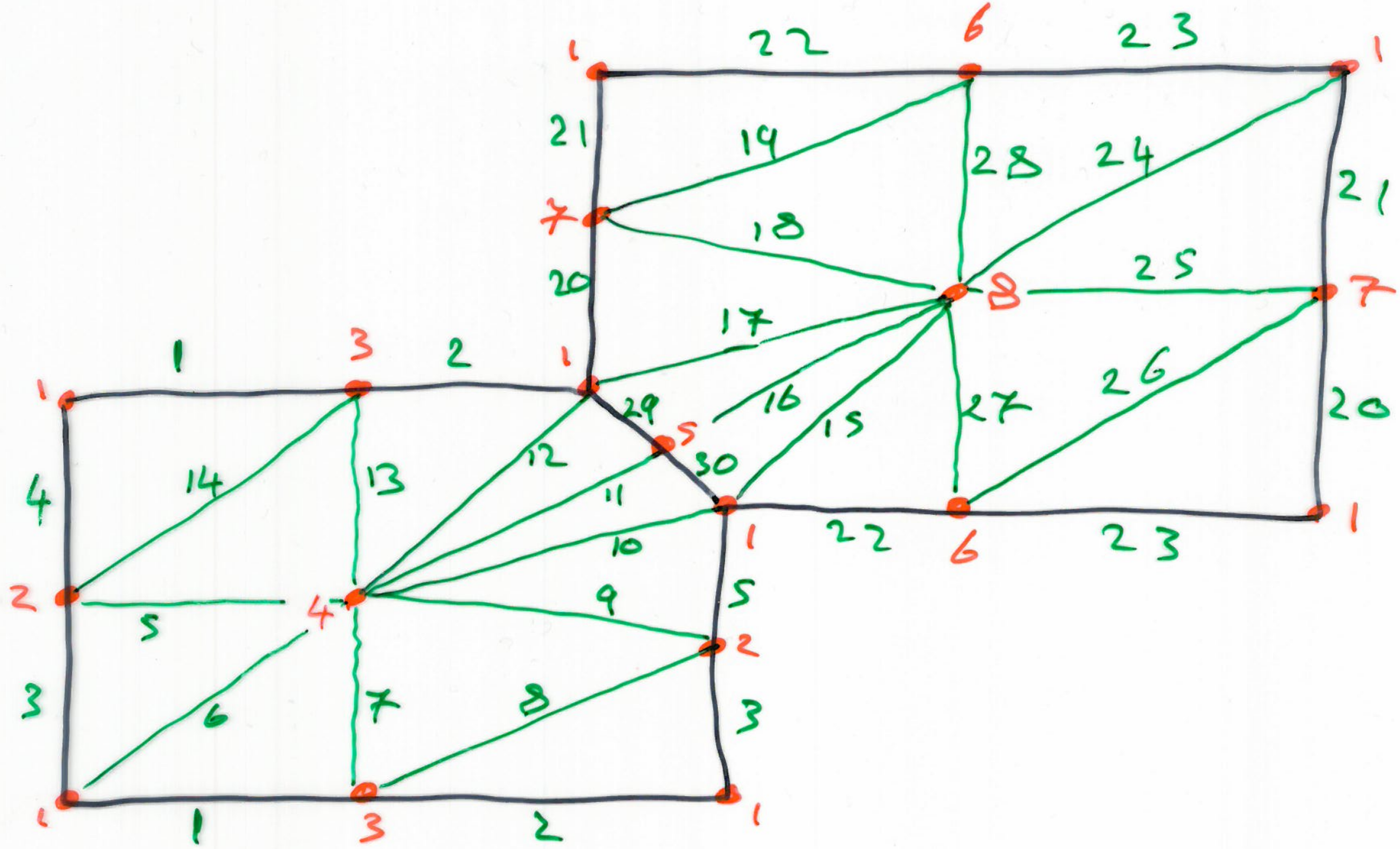


$\chi(\text{Double Torus}) = 8 - 30 + 20 = -2$

Defn Two spaces X, Y are homotopy equivalent if there exist maps

$$f : X \to Y, \quad g : Y \to X$$

with

$$gf \simeq 1_X, \quad fg \simeq 1_Y$$

where $1_X : X \to X$ is the identity map, $1_Y : Y \to Y$ is the identity map on Y.

Example 1 If X and Y are homeomorphic then they are also homotopy equivalent. If $f : X \to Y$ is a homeomorphism then there is a map $g : Y \to X$ with

$$gf = 1_X \quad \text{and} \quad fg = 1_Y.$$

Example 2 $X = \mathbb{C} \setminus \{0\}$

and $Y = S^1$

are homotopy equivalent.

We have

$$f : X \longrightarrow Y = S^1 \, , \quad z \longmapsto \frac{1}{|z|} z$$

$$g : Y = S^1 \longrightarrow X \, , \quad z \longmapsto z \, .$$

Then $fg = 1_Y$, and so $fg \simeq 1_Y$.

To see that $gf \simeq 1_X$ we use the homotopy

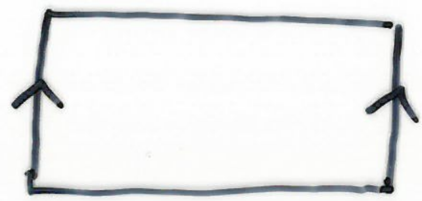
$$H : X \times [0,1] \longrightarrow X \, , \; (z, t) \mapsto \left(\frac{1-t}{|z|} + t \right) z$$

Major Theorem

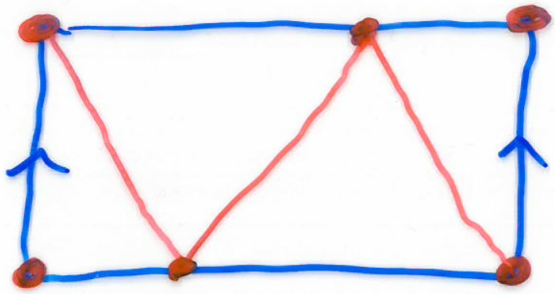
Let X, Y be spaces with triangulations. If X and Y are homotopy equivalent then $\chi(X) = \chi(Y)$.

Proof Beyond scope of this module.

Illustration The circle S^1 is homotopy equivalent to the cylinder

$S^1 \times [0,1] =$  $=$ 

$$\chi(S^1) = 3 - 3 = 0$$



$\chi(\text{cylinder}) = 4 - 8 + 4 = 0$

<u>Illustration</u> The n-simplex Δ^n is homotopy equivalent to a singleton space $\{*\}$

So $\chi(\Delta^n) = \chi(\{*\}) = 1$.

<u>Illustration</u>

$$\chi(S^n) = \chi(\Delta^{n+1}) \pm 1 = 1 \pm 1$$

$$= \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

Recall

$$D^n = \{ x \in \mathbb{E}^n : \|x\| \leq 1 \}$$

<u>Brouwer's</u> <u>Theorem</u> For any continuous map $f : D^n \to D^n$ there exists at least one $x \in D^n$ such that $f(x) = x$.

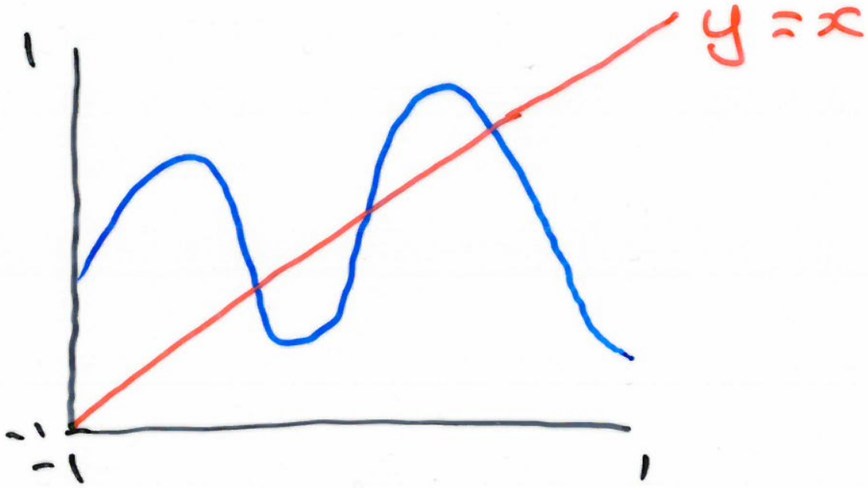
<u>Defn</u> For any map $f : X \to X$ a point $x \in X$ satisfying $f(x) = x$ is called a <u>fixed point</u> of X. So Brouwer's Theorem says that any map $f : D^n \to D^n$ has at least one fixed point.

Brouwer's Theorem for $n=1$

$D^1 = [-1,1]$

We picture a map $f: [-1,1] \to [-1,1]$

by its graph.



A fixed point of f is a point where the blue graph intersects the red line $y=x$.

Brouwer's Theorem says such a point always exists.