we can identify

S^1 and $[0,1]/\{0=1\}$.

Thus maps $f:[0,1]\to S^1$ with $f(0)=f(1)=1$ can be regarded as maps $f: S^1 \to S^1$.

Each map $f:[0,1]\to S^1$ with $f(0)=f(1)=1$ has a winding number $\tilde{f}(1)\in\mathbb{Z}$.

By Proposition 2 last lecture,

if

$$f, g: \underset{\underset{[0,1]/\{0=1\}}{\parallel}}{S^1} \longrightarrow S^1$$

are homotopic then $\tilde{f}, \tilde{g}:[0,1]\to\mathbb{R}$ are homotopic. Furthermore, $\tilde{f}(1)=\tilde{g}(1)$ since the homotopy restricts to a continuous map

$H(1,t)$ of t which is an integer for all $0 \leq t \leq 1$. Thus

Proposition Homotopic maps

$$f, g : S^1 \longrightarrow S^1$$

have the same winding number $\tilde{f}(1) = \tilde{g}(1)$.

We thus get a function

$$\omega : [S^1, S^1] \longrightarrow \mathbb{Z}$$

$$[f] \longmapsto \text{winding number of } f.$$

The function ω is onto because, for any integer n, we can construct a map $f : S^1 \to S^1$ that maps S^1 n times around S^1.

Exercise: $\omega : [S^1, S^1] \to \mathbb{Z}$ is injective (and hence bijective).

Corollary:

<u>Fundamental Theorem of Algebra</u>

Any polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$ and of degree $n > 0$ has at least one zero in $\mathbb{C}$.

<u>proof</u> since $a_n \neq 0$, a scalar multiple of the polynomial has the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

For $\lambda \geq 0$ define the map

$$f_\lambda : S^1 \longrightarrow S^1$$

by

$$f_\lambda(z) = \frac{p(\lambda z)}{|p(\lambda z)|}$$

Any two maps f_λ, $f_{\lambda'}$ are homotopic via the homotopy

$$H_t(z) = \frac{p(((1-t)\lambda + t\lambda')z)}{|p((1-t)\lambda + t\lambda')z|}.$$

Note that $f_0(z)$ is a constant map and thus has winding number 0.

Exercise: For large λ we have $f_\lambda(z)$ is homotopic to the map

$$g : S^1 \to S^1,\ z \mapsto z^n.$$

But $g(z) = z^n$ has winding number $n \neq 0$. This contradiction proves the theorem. $\square$