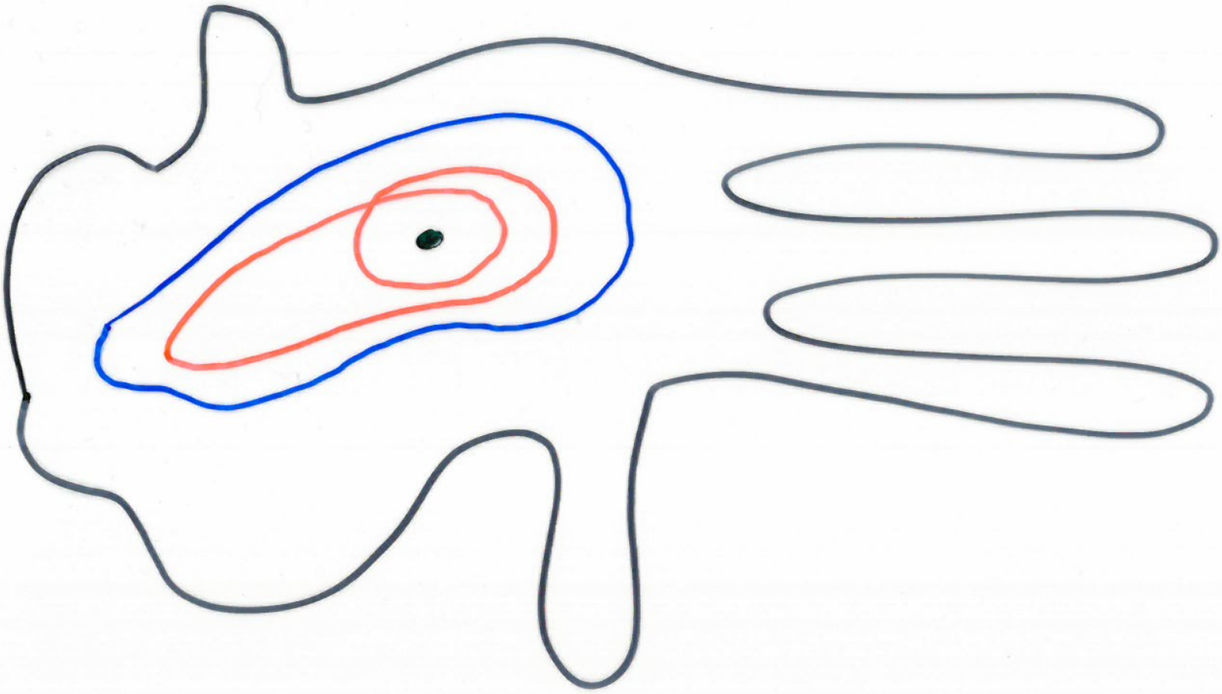
Consider three maps

$f : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$

$g : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$

$h : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$



where f, g are injective, and there are just two values $t_1, t_2 \in S^1$ such that $h(t_1) = h(t_2)$

We can "see" that $f \simeq g$

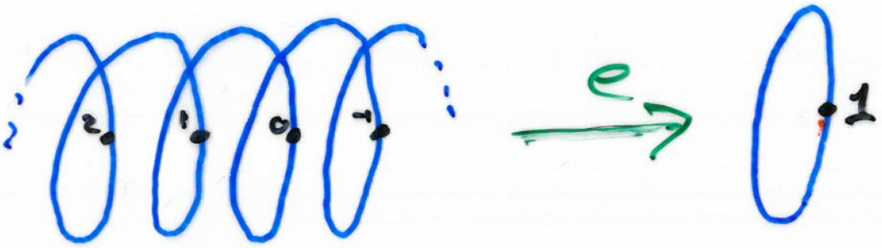
and f is <u>not</u> homotopic to h.

Today: $[S^1, S^1] \cong \mathbb{Z}$

Recall $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

We have the map

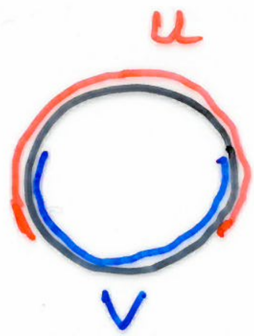
$$e : \mathbb{R} \longrightarrow S^1, \quad \theta \longmapsto e^{2\pi\theta i}$$



<u>Proposition 1</u> Let $f : [0,1] \longrightarrow S^1$ be any continuous map. Then there is a unique continuous map $\tilde{f} : [0,1] \longrightarrow \mathbb{R}$ such that $f(t) = e \circ \tilde{f}(t)$ for $0 \leq t \leq 1$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[ur, dashed, "\exists ! \tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

<u>Proof</u> Consider an open cover of S^1



$$S^1 = U \cup V$$

There is a corresponding open cover of $\mathbb{R}$

$$U_0, V_0, U_1, V_1, U_{-1}, V_{-1}, U_2, V_2, U_{-2} \cdots$$

such that e restricts to homeomorphisms

$$h_l^U : U_l \longrightarrow U, \quad \theta \longmapsto e^{2\pi\theta i}$$

$$h_l^V : V_l \longrightarrow V, \quad \theta \longmapsto e^{2\pi\theta i}$$

for each $l \in \mathbb{Z}$.

<u>Check</u>: for each $t \in [0,1]$ there is an open set

$$t \in W_t \subseteq [0,1]$$

such that

$$f(W_t) \subset U$$

or

$$f(W_t) \subset V.$$

Now $\{W_t\}_{t\in[0,1]}$ is an open cover of $[0,1]$. By compactness of $[0,1]$ there is a finite subcover

$$W_{t_1}, W_{t_2}, \dots, W_{t_k}$$

of $[0,1]$.

We define $\tilde{f}$ on W_{t_1} by

$$\tilde{f}(t) = \begin{cases} (h_0^U)^{-1}(f(t)) & \text{if } f(W_{t_1}) \subset U \\ (h_0^V)^{-1}(f(t)) & \text{if } f(W_{t_1}) \subset V. \end{cases}$$

We inductively define $\tilde{f}$ on each W_{t_i} by similar formulae.

$\boxtimes$

<u>Definition</u> Suppose $f : [0,1] \to S^1$ be any continuous function with $f(0) = f(1) = 1$. Then $\tilde{f}(1)$ is an integer called the <u>winding number</u>

<u>Proposition 2</u> Let $H : [0,1] \times [0,1] \to S^1$ be any continuous map such that $H(0,t) = H(1,t) = 1$ for all $t \in [0,1]$. Then there is a unique map

$\tilde{H} : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ with

$e \circ \tilde{H} = H$ and $\tilde{H}(0,t) = 0$

for all $t \in [0,1]$.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "e"] \\
{[0,1] \times [0,1]} \arrow[r, "H"] \arrow[ur, dashed, "\exists!\, \tilde{H}"] & S^1
\end{tikzcd}$$

<u>proof</u> similar to prop. 1.