Homotopy is a key notion used in the proof that the Euler characteristic of a space is a topological property.

Defn Two maps $f: X \to Y$, $g: X \to Y$ are homotopic if there is a continuous map

$$H : X \times [0,1] \longrightarrow Y \;, \; (x,t) \longmapsto H_t(x)$$

such that $H_0(x) = f(x)$, $H_1(x) = g(x)$ for all $x \in X$.

We can think of $H_t(x)$ as a family of maps $H_t : X \to Y$, $x \mapsto H_t(x)$.

We often refer to H as a homotopy, and write $f \simeq g$.

To fully understand homotopy we need to be clear about the topology on $X \times [0,1]$.

Let $\mathcal{B}$ be the collection of all sets

$$U \times J = \{(u, j) : u \in U, j \in J\}$$

where $U \subseteq X$ is open and $J \subseteq [0,1]$ is open. A set is open in $X \times [0,1]$ if it can be written as a union of sets in $\mathcal{B}$.

Intuitively $H : X \times [0,1] \to Y$, $(x, t) \mapsto H_t(x)$ is continuous if a small change in x, t yields only a small change in $H_t(x)$.

<u>Example</u> Let $Y \subset \mathbb{E}^k$ be a convex set. Let X be any topological space.

Any two maps $f, g : X \to Y$ are homotopic. To see this, define:

$$H : X \times [0,1] \to Y, \quad (x, t) \mapsto f(x) + t(g(x) - f(x))$$

$$= (1-t) f(x) + t g(x)$$

we have $H_t(x) \in Y$ since Y is convex. Also, $H_0(x) = f(x)$, $H_1(x) = g(x)$. One can check that H is continuous.

Proposition For spaces X, Y homotopy is an equivalence relation on continuous maps $X \to Y$.

Proof

For any $f : X \to Y$ we have $f \simeq f$ thanks to the homotopy

$$H_t(x) = f(x).$$

For any $f : X \to Y$, $g : X \to Y$ if $f \simeq g$ then there is a homotopy $H_t(x)$ with $H_0(x) = f(x)$, $H_1(x) = g(x)$. To see that $g \simeq f$ we define

$$H'_t(x) = H_{1-t}(x).$$

For $f, g, h : X \to Y$ if $f \simeq g$ and $g \simeq h$ then there are homotopies

$H_t(x)$, $H_0(x) = f(x)$, $H_1(x) = g(x)$,

$H'_t(x)$, $H'_0(x) = g(x)$, $H'_1(x) = h(x)$.

To see that $f \simeq h$ we define

$$H''_t(x) = \begin{cases} H_{2t}(x) & 0 \leq t \leq \frac{1}{2} \\ H'_{2t-1}(x) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$\boxed{\text{R1}}$

Let $[f]$ denote the homotopy equivalence class of a map $f: X \to Y$. Let $[X, Y]$ denote the set of all homotopy classes $[f]$ where $f: X \to Y$.

There is a vast theory - called homotopy theory - for proving deep results such as:

$$[S^3, S^2] \text{ is bijective with } \mathbb{Z}$$

$$[S^{14}, S^7] \text{ has precisely } 120 \text{ classes}$$

We'll investigate $[S^1, S^1]$.