Simplicial Complexes

Let $v_0, v_1, \dots, v_k$ be vectors in $\mathbb{E}^n$. These vectors are in **general position** if the vectors

$$v_1 - v_0,\quad v_2 - v_0,\quad \dots,\quad v_k - v_0$$

are linearly independent.

Example $v_0 = (1,0,0)$, $v_1 = (0,1,0)$, $v_2 = (0,0,1)$.

Then

$$v_1 - v_0 = (-1,1,0)$$

$$v_2 - v_0 = (-1,0,1)$$

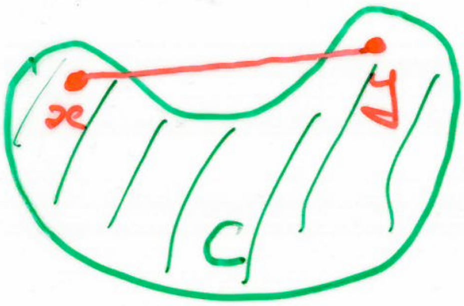
are linearly independent. In other words

$$\lambda_1(v_1 - v_0) + \lambda_2(v_2 - v_0) = (0,0,0)$$

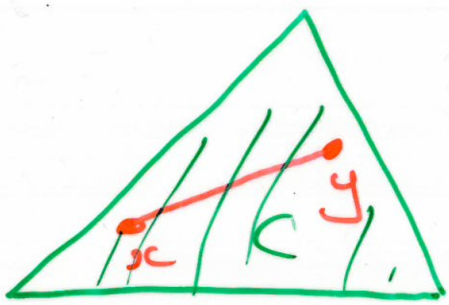
implies $\lambda_1 = \lambda_2 = 0$.

So v_0, v_1, v_2 are in general position.

Recall: A set $C \subseteq \mathbb{E}^n$ is said to be convex if, for any $x, y \in C$, all points on the line from x to y lie in C.



Not Convex



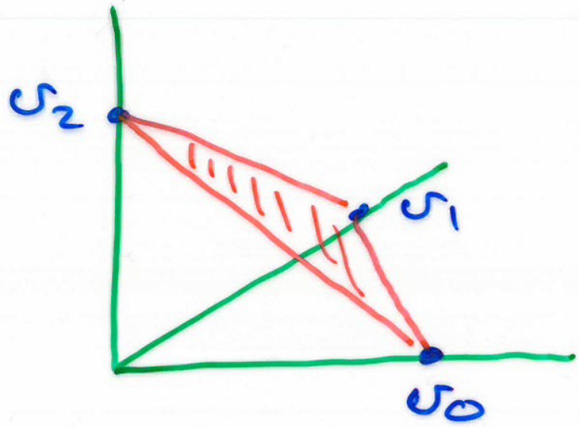
Convex

Suppose $v_0, v_1, \ldots, v_k \in \mathbb{E}^n$ are in general position. Let

$$\text{Convex}(v_0, \ldots, v_k)$$

denote the smallest convex set in $\mathbb{E}^n$ containing $v_0, \ldots, v_k$.

__Example__ $v_0 = (1,0,0)$, $v_1 = (0,1,0)$, $v_2 = (0,0,1)$



In general, $\mathrm{Convex}(v_0, v_1, \dots, v_k)$ consists of all those points of the form

$$x = \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_k v_k$$

with $\lambda_0 \geq 0, \lambda_1 \geq 0, \dots, \lambda_k \geq 0$ and with $\lambda_0 + \lambda_1 + \dots + \lambda_k = 1$.

__Defn__ Let $v_0, v_1, \dots, v_k \in \mathbb{E}^n$ be in general position, we call

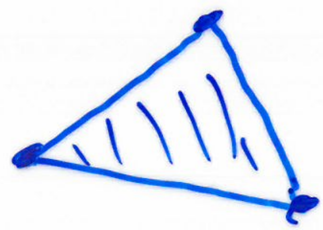
$$\mathrm{Convex}(v_0, \dots, v_k)$$

a __simplex__ of dimension k, or __k-simplex__.

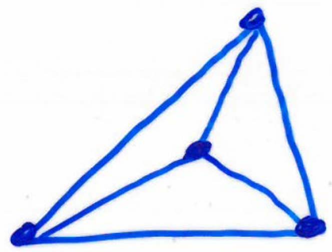
0-simplex = point

1-simplex = line segment

2-simplex = solid triangle



3-simplex = tetrahedron



Simplexes have "faces".

If A and B are simplexes, and if the vertices of A form a subset of the vertices of B, then we say that A is a face of B.

Example A 3-simplex has:

four faces of dimension 2

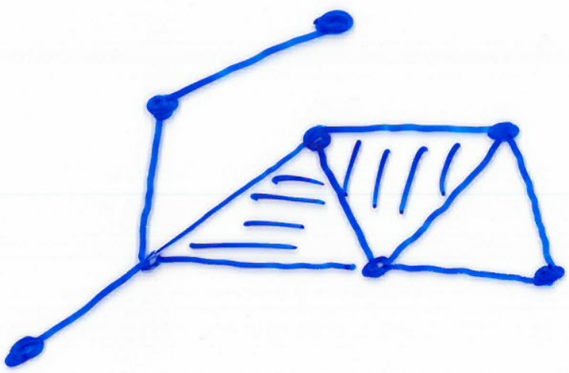
six faces of dimension 1

four faces of dimension 0.

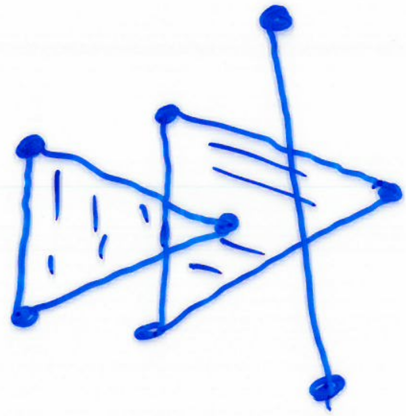
Definition A finite collection of simplexes in $\mathbb{E}^n$ is called a simplicial complex if;

i) whenever a simplex lies in the collection then so too does all of its faces;

ii) whenever two simplexes of the collection intersect, they do so in a common face.



simplicial complex



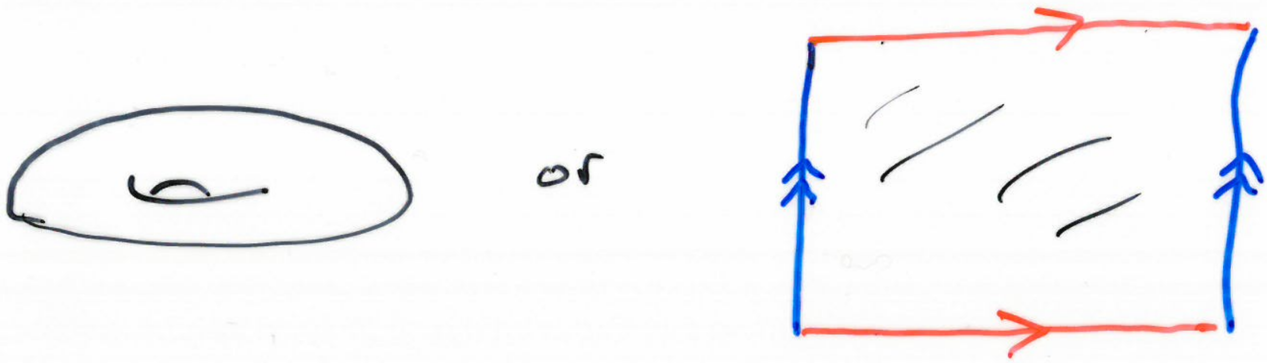
Not a simplicial complex

A simplicial complex is a subset of Euclidean space, and as such is a topological space.

We write $K, L, \ldots$ for simplicial complexes. We write $|K|, |L|$ for the corresponding topological subspaces of $\mathbb{E}^n$.

Defn A triangulation of a topological space X consists of a simplicial complex K and homeomorphism $h : |K| \to X$.

Example



We can build a simplicial complex K homeomorphic to the torus :