Compactness is a topological property because:

Proposition Suppose $f: X \to Y$ is a continuous map. If X is compact then so too is the image $f(X)$.

Proof Let $\mathcal{F}$ be an open cover of the image $f(X)$. Then

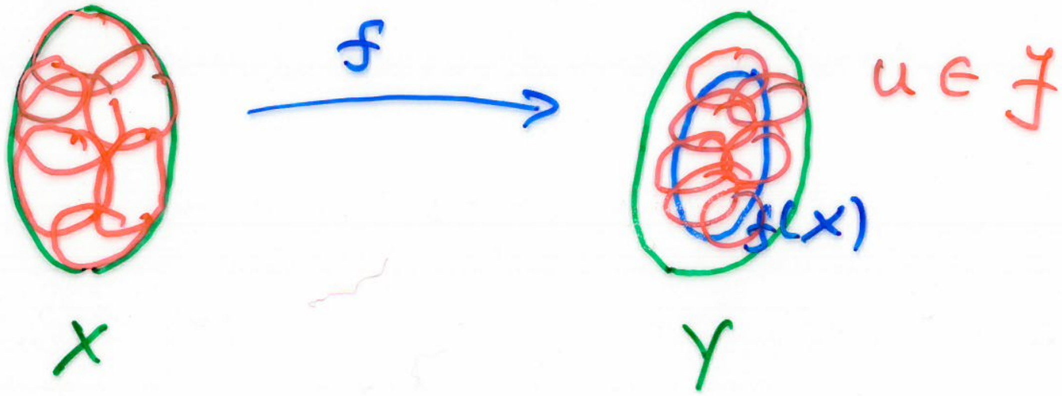
$\{f^{-1}U\}_{U \in \mathcal{F}}$ is an open cover

of X. If X is compact then there is a finite subcover

$\{f^{-1}U\}_{U \in \mathcal{F}'}$ of X. But then

$\{U\}_{U \in \mathcal{F}'}$ is a finite open cover of $f(X)$.

Hence $f(X)$ is compact.



<u>Defn</u> Let X be a topological space. A subset $A \subseteq X$ is <u>closed</u> if its complement $X \setminus A$ is an open subset of X.

<u>Example</u> $[0,1] \subseteq \mathbb{R}$ is closed since $\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$ is open.

Example $(0,1] \subseteq \mathbb{R}$ is neither open nor closed.

Defn Let A be a subset of the space X. A point $p \in X$ is an accumulation point of A if every open set $U \subset X$ that contains p also contains some point in $A \setminus \{p\}$.

Example Let $X = \mathbb{R}$, $A = \left\{ \frac{1}{n} \right\}_{n=1,2,3,\dots}$. In this example 0 is (the only) accumulation point of A.

Example Let $X = \mathbb{R}$, $A = [0, 1)$.

Then every point in A is an accumulation point. So too is 1.

Proposition A set $A \subseteq X$ is closed iff A contains all its accumulation points.

Proof If A is closed then $X \setminus A$ is open. So no point of $X \setminus A$ is an accumulation point of A. So A contains its accumulation points.

Conversely, suppose A contains all its accumulation points.

Let $x \in X \setminus A$. We can find an open set $U \subseteq X \setminus A$ such that $x \in U$, since x is not an accumulation point. So $X \setminus A$ is open. So A is closed.

$\blacksquare$

Theorem A compact subset of a Euclidean space $\mathbb{E}^k$ is closed and bounded.

Proof See Theorem 3.9 in the book.

<u>Recall</u> Our continuous space filling curve $f : [0,1] \longrightarrow \Delta$.

It is easy to see that each point in Δ is an accumulation point of $f([0,1])$.

Now $[0,1]$ is compact, so by first proposition today, $f([0,1])$ is compact, so, by the above theorem, $f([0,1])$ is closed. So $f([0,1])$ contains all its accumulation points. Thus f is surjective •