Compactness

Let X be a topological space. Let $\mathcal{F}$ be a family of open subsets of X whose union equals X. Then we say that $\mathcal{F}$ is an open cover of X.

If $\mathcal{F}'$ is a subfamily of $\mathcal{F}$ and if the union of all sets in $\mathcal{F}'$ equals X then we say that $\mathcal{F}'$ is a subcover of the open cover $\mathcal{F}$.

Example $X = \mathbb{R}$. Let

$$\mathcal{F} = \{(n-2, n+2)\}_{n \in \mathbb{Z}}.$$

Then $\mathcal{F}$ is an open cover of $X = \mathbb{R}$.

Let $2\mathbb{Z}$ be the even integers.

Then

$$\mathcal{F}' = \{(n-2, n+2)\}_{n \in 2\mathbb{Z}}$$

is a subcover of $X = \mathbb{R}$.

An open cover $\mathcal{F}$ of X is said to be <u>finite</u> if $\mathcal{F}$ consists of only finitely many open sets.

<u>Example</u> $X = \mathbb{R}$.

$$\mathcal{F} = \{(-\infty, 2), (-2, 2), (0, \infty)\}.$$

This $\mathcal{F}$ is a finite open cover of $X = \mathbb{R}$.

Definition A topological space X is compact if every open cover of X has a finite subcover.

Example The space $X = \mathbb{R}$ is not compact. To see this we just note that the open cover

$$\mathcal{F} = \{(n-1, n+1)\}_{n \in \mathbb{Z}}$$

has no finite subcover of $\mathbb{R}$.

Theorem The closed interval $[a, b]$ in $\mathbb{R}$ is compact.

<u>Proof</u> Let $\mathcal{F}$ be an open cover of $[a,b]$. Define a subset $X \subseteq [a,b]$ by:

$$X = \{ x \in [a,b] : [a,x] \text{ is contained in a finite subcover of } \mathcal{F} \}.$$

Note that X is non-empty since $a \in X$.

Note that X is bounded above by b. Thus X has a least upper bound s.

We need to prove $s = b$.

Let $U \in \mathcal{F}$ that contains s.

Since U is open we can choose $\varepsilon > 0$ such that $(s-\varepsilon, s] \subseteq U$;

if $s < b$ then can assume

$(s-\varepsilon, s+\varepsilon) \subseteq U$.

Note: if $x \in X$ and if

$a \leq y \leq x$ then $y \in X$.

Since s is the least upper bound of X, we can assume $s - \varepsilon/2 \in X$.

This means that $[a, s-\frac{\varepsilon}{2}]$ is contained in the union of some finite subcover $\mathcal{F}'$ of $\mathcal{F}$.

Adding U to $\mathcal{F}'$ gives a finite subcover whose union contains $[a, s]$.

Therefore $s \in X$.

If $s < b$ then

$\cup \mathcal{F}' \cup \mathcal{U}$ contains $[a, s + \varepsilon/2]$

which contradicts the fact

that s is the l.u.b. of X.

Hence $s = b$. $\blacksquare$

<u>Theorem</u> Suppose that X is homeomorphic to Y. Then X is compact if and only if Y is compact.

<u>Proof</u> next time.

<u>Example</u> $[a, b]$ is <u>not</u> homeomorphic to $\mathbb{R}$, since

$[0,1]$ is compact and $\mathbb{R}$ is not compact.