Consider

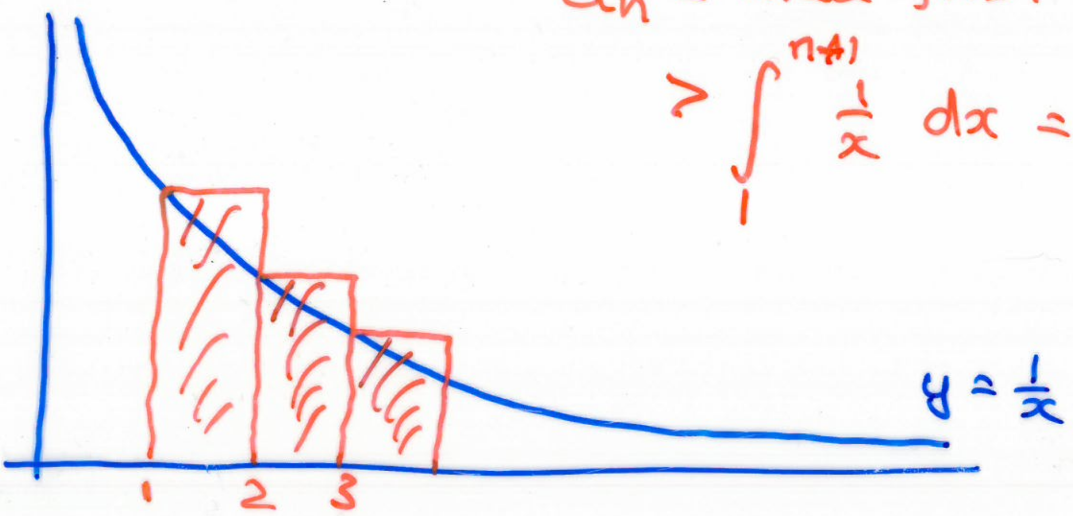
$a_1 = 1,\ a_2 = 1\frac{1}{2},\ a_3 = 1\frac{5}{6},\ \dots,\ a_n = a_{n-1} + \frac{1}{n}$

So $\lim\limits_{n\to\infty} a_n - a_{n-1} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

$\lim\limits_{n\to\infty} a_n =$ does not exist

$a_n =$ area first n boxes

$> \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$



<u>Defn</u> A sequence of points $a_1, a_2, a_3, \dots$ in $\mathbb{E}^k$ is said to be a <u>Cauchy</u> <u>sequence</u> if for any $\varepsilon > 0$ there is an N

such that

$$\| a_m - a_n \| < \varepsilon$$

for all $m, n \geq N$.

<u>Theorem</u> Any Cauchy sequence $a_1, a_2, \ldots$ in $\mathbb{E}^k$ has a limit

$$\lim_{n \to \infty} a_n .$$

<u>Example</u>

$a_1 = 1$, $a_2 = 1\frac{1}{2}$, $a_3 = 1\frac{1}{4}$, $a_n = a_{n-1} + \frac{1}{2^n}$.

For $m > n$ we have

$$a_m - a_n = \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^{m-n}}\right)$$

$$< \frac{1}{2^n} .$$

So for any $\varepsilon > 0$ we can choose N such that $\frac{1}{2^N} < \varepsilon$.

Then $|a_m - a_n| < \varepsilon$

for all $m, n \geq N$. Hence the

sequence is Cauchy.

Check: $\lim_{n \to \infty} a_n = 2$.

Last lecture: we constructed

functions

$f_1 : [0,1] \to \Delta$, $f_2 : [0,1] \to \Delta$, ...

Fix any $t \in [0,1]$.

For any $m > n$ we have

that $f_m(t)$ and $f_n(t)$ both

lie in a single equilateral

triangle of side $\frac{1}{2^n}$.

So the sequence

$f_1(t), f_2(t), f_3(t), \dots$ in $\mathbb{E}^2$

is Cauchy. The sequence thus has a limit which we denote by $f(t)$.

Furthermore, if t is close to t' then $f(t)$ is close to $f(t')$. "Thus" $f : [0,1] \to \Delta$ is continuous.

It remains to prove that $f(t)$ is surjective. (For this we need "compactness".)

Let's aim for a proof that $[0,1]$ is <u>not</u> homeomorphic to Δ.

<u>Proposition</u> Let $f : X \to Y$ be a homeomorphism. Then X is connected if and only if Y is connected.

<u>Proof</u>

Suppose Y is not connected. Then there are open sets U, V in Y with $U \neq \emptyset \neq V$ and $U \cup V = Y$. Since f is continuous we see that $f^{-1}(U)$ and $f^{-1}(V)$ are open in X.

Let $g: Y \to X$ be such that

$g(f(x)) = x$, $f(g(y)) = y$ for

$x \in X$, $y \in Y$.

so $x = f^{-1}(fx)$. Hence

$$f^{-1}U \cup f^{-1}V = X$$

and $f^{-1}U \neq \phi$ $f^{-1}V$.

Thus X is the union of two non-empty open subsets. Thus X is not connected.

By swapping the roles of f and g we see that if X is not connected then Y is not connected. $\boxed{\checkmark}$

Theorem $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Sketch Proof

Suppose there were a homeomorphism

$$f : \mathbb{R} \to \mathbb{R}^2 .$$

Consider

$$X = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$$

$$Y = \mathbb{R}^2 \setminus \{f(0)\}$$

Exercise: The function

$$f' : X \to Y, \quad x \longmapsto f(x) .$$

is a homeomorphism.

But X is <u>not</u> connected while Y <u>is</u> connected.

This contradicts the above proposition.

Hence no homeomorphism $f: \mathbb{R} \to \mathbb{R}^2$ can exist. $\boxtimes$