Term Structure of Interest Rates

1. Discrete Time Rates

In practice i and δ vary according to the term of an investment.

Definitions

y_n = n year spot rate of interest

= yield on a unit zero coupon bond with term n years.

P_n = price at issue of a unit zero coupon bond maturing in n years.

$$\boxed{P_n (1+y_n)^n = 1}$$

Example

Calculate the price of a five year fixed interest security, redeemable at par, with 6% annual coupons. Assume the annual term structure of interest rates is:

$y_1 = 7\%$ $y_2 = 7\frac{1}{4}\%$ $y_3 = 7\frac{1}{2}\%$ $y_4 = 7\frac{3}{4}\%$ $y_5 = 8\%$

Per €100

$$A = 6\left(v_1 + v_2^2 + v_3^3 + v_4^4 + v_5^5\right) + 100\, v_5^5$$

where $v_1 = \frac{1}{1+0.07}$ $\quad v_2 = \frac{1}{1+0.0725}$ $\quad \ldots$

$$A = 6\,(4.0314) + 68.06 = €\,92.25$$

<u>Definition</u>

$f_{t,r}$ = annual interest rate agreed at time 0 for an investment made at time t for a period of r year

$$\boxed{(1+y_t)^t\,(1+f_{t,r})^r = (1+y_{t+r})^{t+r}}$$

and

$$\boxed{(1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}}}$$

2. Continuous Time Rates

<u>Definition</u>

Y_t : t year spot force of interest

$$\boxed{e^{Y_t} = 1 + y_t}$$

Definition

$F_{t,r}$ = force of interest equivalent to the annual forward rate of interest $f_{t,r}$.

$$\boxed{e^{F_{t,r}} = 1 + f_{t,r}}$$

Example

Calculate the price of 5-year fixed interest security redeemable at par with 9% annual coupons given the term structure:

$Y_1 = 10\%$

$y_2 = 5\%$

$F_{2,1} = 6\%$

$f_{3,1} = 9\%$

$F_{2,3} = 10\%$

Sol^n

$$\text{Price} = 9\left(e^{-Y_1} + \cancel{e}\left(\frac{1}{1+y_2}\right)^2 + e^{-F_{2,1}}\frac{1}{(1+y_2)^2} + \frac{e^{-F_{2,1}}}{(1+y_2)^2(1+f_{3,1})}\right) + 109\, e^{-F_{2,3}}\frac{1}{(1+y_2)^2}$$

$$= €116.796 \quad (\text{per } €100)$$

Relationship between forward rates and zero coupon bond prices.

$$e^{Y_t t} e^{r F_{t,r}} = e^{(t+r) Y_{t+r}}$$

$$\Rightarrow \quad t Y_t + r F_{t,r} = (t+r) Y_{t+r}$$

$$\Rightarrow \quad F_{t,r} = \frac{(t+r) Y_{t+r} - t Y_t}{r}$$

$$\boxed{P_t e^{tY_t} = 1 \qquad Y_t = -\frac{1}{t} \log P_t}$$

$$\Rightarrow \quad \boxed{F_{t,r} = \frac{1}{r} \log \frac{P_t}{P_{t+r}}}$$

<u>Example</u>
The prices for zero coupon bonds of various terms are as follows:

$\overline{1|} = €94\%$ $\quad \overline{5|} = €70\%$ $\quad \overline{10|} = 47\%$ $\quad \overline{15|} = €30$

Calculate Y_{10} and $F_{5,10}$.

<u>Sol^n</u> $\quad Y_{10} = -\frac{1}{10} \log P_{10} = -\frac{1}{10} \log(0.47) = 7.55\%$

$$F_{5,10} = \frac{1}{10} \log\left(\frac{P_5}{P_{15}}\right) = \frac{1}{10} \log\left(\frac{0.7}{0.3}\right) = 8.47\%$$

<u>Definition</u>

$$F_t = \lim_{r \to 0} F_{t,r}$$

we call F_t the instantaneous forward rate at time t.

Note! $$F_t = \lim_{r \to 0} \frac{1}{r} \log\left(\frac{P_t}{P_{t,r}}\right)$$

$$= \lim_{r \to 0} \frac{\log P_t - \log P_{t+r}}{r}$$

$$= -\frac{d}{dt} \log P_t$$

$$\boxed{F_t = -\frac{1}{P_t} \frac{d}{dt} P_t}$$

<u>Example</u> The price at time 0 of a zero coupon bond of term $2 < t < 4$ is given by the equation

$$P_t = (100 - 2t^2)\ \% .$$

Calculate F_3.

<u>Sol^n</u> $$F_t = -\frac{1}{P_t} \frac{d}{dt} P_t = -\frac{1}{100-2t^2}(-4t)$$

$$F_3 = \frac{1}{100-18}(-4 \times 3) = 14 \cdot 6\ \%$$

Yield to maturity / redemption yield

= the effective rate of interest i at which the discounted value of the proceeds of a bond equal the market price.

Example

The current annual term structure of interest rates is : (6%, 6%, 6%, 6%, 7%)
Calculate the gross redemption yield of a 5-year fixed interest security redeemable at par if the coupon is 2% per annum.

Sol^n $\quad P = 2a_{\overline{5}|} + 100v^5$, $\quad i = $ g.r.y.

Also $\quad P = 2a_{\overline{4}|@6\%} + 100v^5_{@7\%} = 79.66$

$\therefore \quad 79.66 = 2\dfrac{(v - v^5)}{1-v} + 100v^5 \qquad i = 6.92\%$

Par Yields

yc_n = n-year par yield
is defined by

$$\boxed{1 = (yc_n) \times (v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n)}$$

<u>Example</u>

Calculate the 3-year par yield if interest rates are: $y_1 = 5\%$, $y_2 = 4\%$, $Y_3 = 3\%$

<u>Sol^n</u>

$$1 = yc_3 \left(\frac{1}{1.05} + \frac{1}{(1.04)^2} + e^{-0.09} \right)$$

$$\Rightarrow yc_3 = \text{etc.}$$

Duration, Convexity & Immunisation

Interest Rate Risk

Institution has assets of NPV V_A
" " liability " " V_L

We hope $V_A \geq V_L$.

Note that V_A, V_L depend on the effective rate i.

If i changes will $V_A \geq V_L$ still hold?

Effective duration / Volatility

This is a measure of the sensitivity of the values of a series of cash flows to movements in the rates of interest.

$$A = \sum_{k=1}^{n} C_{t_k} v^{t_k}$$

C_{t_k} = cashflow at time t_k

$v = \frac{1}{1+i}$ depends on i

Defn Effective duration or volatility is

$$\nu(i) = -\frac{1}{A}\frac{d}{di}A$$

Thus

$$\nu(i) = \frac{1}{A} \sum_{k=1}^{n} \left(C_{t_k} t_k v^{t_k+1} \right)$$

<u>Remark</u> This definition of $\nu(i)$ assumes the cashflows are independent of i.

<u>Duration</u>

Another measure of interest rate sensitivity.

<u>Defn</u> Duration is

$$\tau(i) = (1+i)\,\nu(i)$$

Thus

$$\tau(i) = \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

<u>Example</u> Find the duration of an n-year zero coupon bond redeemable at R.

<u>Soln</u>

$$\tau(i) = \frac{n R v^n}{R v^n} = n$$

<u>Example</u> Find the duration of an n-year bond paying coupons of D annually in arrears.

<u>Sol</u>

$$\tau(i) = \frac{D(Ia)_{\overline{n}|} + Rnv^n}{Da_{\overline{n}|} + Rv^n}$$

<u>Example</u> Evaluate the duration of a bond redeemable at par in 10 years time with annual coupons of 8% at interest rates $i = 5\%, 10\%, 15\%$

<u>Sol</u>

$$\tau(i) = \frac{8(Ia)_{\overline{10}|} + 100(10)v^{10}}{8a_{\overline{10}|} + 100v^{10}}$$

$$(Ia)_{\overline{10}|} = \frac{\ddot{a}_{\overline{10}|} + 10v^{10}}{i}$$

$\tau(5) = 7.54$

$\tau(10) = 7.04$

$\tau(15) = 6.52$

N.B.
Large $\tau \Rightarrow$ sensitive to interest rate movements

Convexity

$$C(i) = \frac{1}{A}\frac{d^2}{di^2}A$$

Then

$$C(i) = \frac{\sum_{k=1}^{n} C_{t_k} t_k (t_k+1) v^{t_k+2}}{\sum_{k=1}^{n} C_{t_k} v^{t_k}}$$

Note: Convexity is a measure of the change in duration when the interest rate changes

General problem

Estimate $\dfrac{A(i+\varepsilon) - A(i)}{A(i)}$.

Well: Taylor's expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

yields

$$\frac{A(i+\varepsilon) - A(i)}{A} \simeq -\varepsilon V(i) + \frac{1}{2}\varepsilon^2 C(i)$$

Example Calculate, using 10% interest, the convexity of:

a) 11-year zero coupon bond

b) a lump sum of £9,665 in 5 year's time and a lump sum payment of £26 910 in 20 years time

c) a level perpetuity annually in arrear.

a) $P_A(i) = 100v^{11} = 100(1+i)^{-11}$

$$c_A(i) = \frac{P''_A(i)}{P_A(i)} = \frac{100(-11)(-12)(1+i)^{-13}}{100(1+i)^{-11}}$$

$$= \frac{132}{(1+i)^2} = 109.1$$

b) $P_B = 9665(1+i)^{-5} + 26910(1+i)^{-20}$

$$c_B(i) = \frac{P_B''(i)}{P_B(i)} = \ldots - 153.7$$

c) $P_c(i) = a_{\overline{\infty|}} = \frac{1-v^{\infty}}{i} = \frac{1}{i}$

$$c_c(i) = \frac{P''_c(i)}{P_c(i)} = \frac{(-1)(-2)i^{-3}}{i^{-1}} = \frac{2}{i^2}$$

$c_c(i) = 200$

Immunisation

Suppose an organisation has liabilities involving a known series of cashflows.

Suppose it has asset cashflows that exactly match (in time and amount) the liabilities.

The organisation is <u>perfectly</u> <u>immunised</u> against interest rate changes.

In practice perfect immunisation is rarely possible.

Remingtons Conditions for Immunisation

① $V_A(i_0) = V_L(i_0)$ → the value of the assets equals the value of the liabilities at the starting interest rate i_0

② $\nu_A(i) = \nu_L(i)$ → the volatilities of the asset and ~~the~~ liabilities are equal for all i.

3 $C_A(i) > C_L(i)$ $\rightarrow$ the convexity of assets is greater than convexity of liabilities, all i.

Under conditions 1, 2 & 3 consider

$$S(i) = V_A(i) - V_L(i).$$

$$S(i_0 + \varepsilon) \simeq S(i_0) + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2} S''(i_0)$$

$$= \underset{1}{0} + \underset{2}{0} + \underset{3}{>0}$$

So $S(i) > 0$ for i near i_0.