Actuarial Mathematics (MA310)

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Section B: Project Appraisal

Methods that can be used to decide between alternative investment projects:

- NPV and Accumulated Profit
- Internal Rate of Return
- Payback period
- Discounted Payback Period

Measures of investment return:

- Money Weighted Rate of Return
- Time Weighted Rate of Return
- Limited Internal Rate of Return

Cashflow Model

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NET CASHFLOW
Discrete: $c_t =$ (cash inflow @t) - (cash outflow @t)
Continuous: $\rho(t) = \rho_1(t)$ - $\rho_2(t)$
$\rho(t) =$ net rate of CF per unit of time at time t

Accumulated Profit

$$A(T) = \sum c_t(1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t}\,dt$$

T = Time T when project ends
i = Effective ROI per unit of time
Appraisal: Choose project with highest $A(T)$

Net Present Value

$$NPV(i) = \sum c_t(1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t}\,dt$$

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$$A(T) = NPV(i) \times (1+i)^T$$

Example

A software company is to set up a new computer system for a client and can choose between contracting out the work (Proj A) and doing it all in-house (Proj B).

	Description	Timing	Amount
Project A	Contractor fees	Start Y1	$< 150,000 >$
	Contractor fees	Start Y2	$< 250,000 >$
	Contractor fees	Start Y3	$< 250,000 >$
	Sales	End Y3	$1000,000$
Project B	New equipment	Start Y1	$< 325,000 >$
	Staff costs	Through Y1	$< 75,000 >$
	Staff costs	Through Y2	$< 90,000 >$
	Staff costs	Through Y3	$< 120,000 >$
	Sales	End Y3	$1000,000$

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- Conclusion: @ 20% RDR Project A is preferable

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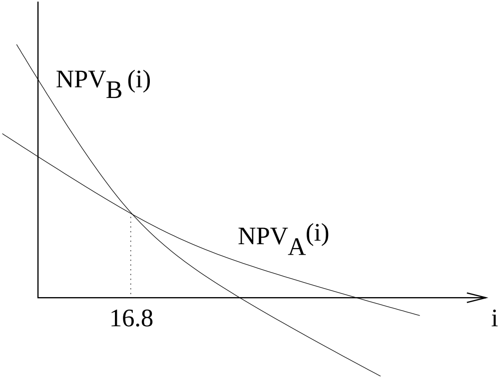

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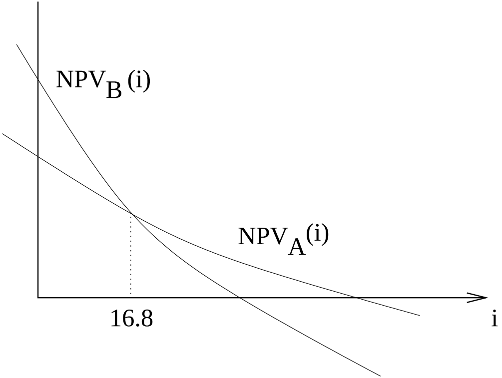

In general: $NPV(i_1) > 0 \Rightarrow$ project profitable @ROI i_1

Example

A project has an initial outlay on 1/5/2006 of £4,000,000 . Three months later a further expenditure of £2,000,000 will be required. On 1/10/2007 income will be received of £20,000 a month payable in arrears for 25 years. The income increases by 5% per annum compound on 1st October each year; the first increase of 5% occurs on 1st October 2010.

Calculate the net present value of the project at a rate of 5% per annum.

$v = 1/1.05$

$$NPV = <4000> + <2000> v^{\frac{3}{12}} + (12)(20) v^{1\frac{5}{12}} a^{(12)}_{\overline{3}|}$$

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$$NPV = < 998,800 >$$

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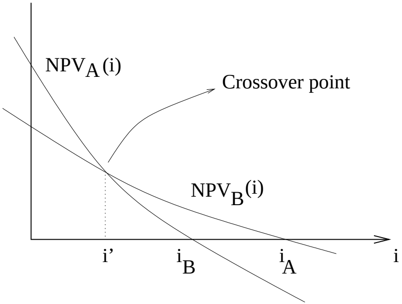
  $NPV_B(0.2) = 40.4$, $NPV_B(.25) = <13.59>$. So $0.2 < \text{IRR} < 0.25$
  Linear Interpolation: IRR = 23.7% approx.

Comparison of two projects



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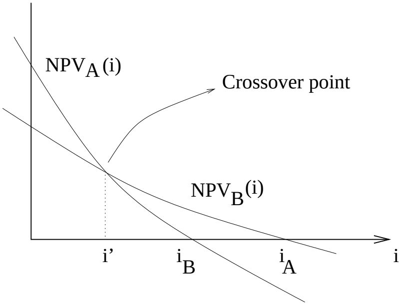

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(A) Select project with higher yield. Choose Project A if $i_A > i_B$

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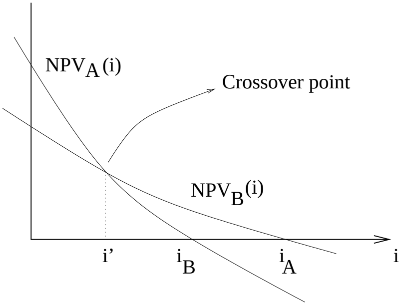

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(A) Select project with higher yield. Choose Project A if $i_A > i_B$

(B) Select project with higher $NPV(i_1)$ (= profit), where i_1 is the ROI at which funds are borrowed or lent.
Choose Project A if $i_1 > i'$. Choose Project B if $i_1 < i'$.

Criteria (A) and (B) don't always agree.
Optimal criteria: Use (B) but, before recommending project, check it gives a profit.

Example

An investor is considering whether to invest in either/both of the following loans.

- Loan X: For a purchase price of £10,000 the investor will receive £1,000 pa payable quaterly in arrears for $\overline{15|}$.
- Loan Y: For a purchase price of £11,000 the investor will receive an income of £605 pa, payable annually in arrears for $\overline{18|}$, and a return of his outlay at the end of this period.

The investor may borrow or lend money at 4% pa .

Would you advise the investor to invest in either loan? If so, which would be the more profitable?

Loan X:

$$NPV_X(i) =< 10,000 > +1000\, a^{(4)}_{\overline{15}|}$$

$$=< 10,000 > +(1 - v^n)/4((1+i)^{\frac{1}{4}} - 1)$$

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$NPV_X(.04) = £1,284$

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$NPV_Y(i_Y) = 0$ when $i_Y = 5.5\%$

$NPV_Y(.04) = £2{,}089$

Summary of results:

Loan	Yield	NPV@4%
X	5.99%	1284
Y	5.5%	2089

Both are profitable. Do we choose X or Y?

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Note: The ROI at which investor lends/borrows which influences the selection.

Different ROI for lending and borrowing

I reality an investor pays j_1 on borrowings and receives j_2 on investments ($j_1 > j_2$).

So the ROI depends on whether or not the investor's account is in credit.

Example

	Outlay		Proceeds	
Project D	@Start	100,000	end $\overline{5}\vert$	140,000
Project E	@Start	80,000	end Y1	10,000
	Start Y2	20,000	end Y2	30,000
	Start Y3	5,000	end Y3	57,000

A company is to choose between D and E, both of which would be financed by a loan, repayable only at the end of the project. The company must pay 6.25% pa on money borrowed, but can earn only 4% on money invested in its deposit account.

Project D:
Accumulated Profit at end of $\overline{5}|$ =

$$< 100,000 > \times (1.0625)^5 + 140,000 = 4,592$$

Project E Cashflows:

@ start: $< 80,000 >$

@ end Y1:

$.0625 < 80,000 > +10,000+ < 20,000 >=< 15,000 >$

@ end Y2: $.0625 < 95,000 > +30,000+ < 5,000 >= 19,062.5$

@ end Y3: $.0625 < 95,000 > +87,000 + .04 \times 19,062.5 = 81,825$

Then $81,825+ < 95,000 > +19,062.5 = 5887.50$

@ end Y5: $5887.5 \times (1.04)^2 = 6368$

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@ end Y5: $5887.5 \times (1.04)^2 = 6368$

Conclusion: Project E is preferable (higher profit)

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DPP is the smallest value of t such that $A(t) \geq 0$ where

$$A(t) = \sum_{s<t} c_s(1+j_1)^{t-s} + \int_0^t \rho(s)(1+j_1)^{t-s}\, ds$$

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If DPP$=t_1$ then

$$A(T) = A(t_1)(1+j_2)^{T-t_1} + \sum_{t>t_1} c_t(1+j_2)^{T-t} + \int_{t_1}^T \rho(t)(1+j_2)^{T-t}\, dt$$

DPP assumption: $A(t)$ changes sign only once.

Appraisal: A project with a shorter DPP is preferable to one with longer DPP because it will produce profits earlier.

Payback Period
$=$ DPP with $j_1 = 0$.

Do NOT use Payback Period.

Example

The business plan for a new company that has obtained a $\overline{5|}$ lease for operating a local bus service is shown below.

Cashflow	Timing	Amount (£000s)
Initial set up costs	@start	$< 100 >$
Advertizing for income	1 month	$+200$
Purchase of vehicles	3 months	$< 2000 >$
Passenger fares	Continuous from 3 months	$+1000$ pa
Staff costs	Continuous from 3 months	$< 400 >$ pa
Resale value of assets	5 years	$+500$

Calculate DPP for the project assuming it will be financed by a flexible loan facility based on an effective ROI of 10% pa .

$$A(t) = < 100 > (1.1)^t + 200(1.1)^{t-\frac{1}{12}} - 2000(1.1)^{t-\frac{3}{12}}$$

$$+ \int_{\frac{3}{12}}^{t} (1000 - 400)(1.1)^{t-s}\, ds$$

$$A(t) = < 1899.2 > (1.1)^{t-\frac{3}{12}} + 600\overline{s}^{10\%}_{\overline{t-\frac{3}{12}}|}$$

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$$A(t) = 0 \Rightarrow \frac{1 - v^{t-\frac{3}{12}}}{\delta} = 3.1654$$

$$t = 4.05$$

Example

A company is considering investing in one of a number of projects. The chosen project is to be financed by a bank loan of £500,000 at an effective rate of interest of 9% per annum. Any surplus funds may be invested at 6% per annum effective. One of the projects has an outlay of £500,000 and income of £70,000 per annum payable continuously for 20 years.

(a) Calculate the discount payback period for the project.

(b) Calculate the accumulated profit after 20 years.

$$A(t) = <500> (1+i)^t + 70\overline{s}_{\overline{t}|}$$

Must Solve $A(t) = 0$ or

$$\frac{500}{70} = \overline{a}_{\overline{t}|} = (1 - (\frac{1}{1.09})^t)/\ln(1.09)$$

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$$t = 11.093$$

$$A(20) = 70,000 \times \bar{s}_{\overline{20-11.093}|@.06}$$

$$= 70,000 \times \frac{(1.06)^{20-11.093} - 1}{\ln(1.06)}$$

$A(20) = £817,320$

Project Appraisal: other considerations

	Initial investment	Proceeds	Yield	Profit @ $\overline{1}|$
Project A	£1	£200 @$\overline{1}|$	199%	£199
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Which project is preferable?

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  e.g. if borrowing ROI=18% then Profit of B = £200. This is greater than the profit on A but maybe prefer A as it's less hassle.

Conclusion: There is more to consider in arriving at a decision that just a comparison of profit.

Cashflow

- Are CF requirements for the project consistent with the business's other needs. (e.g. Could it need future borrowing?)
- Over what period will the profit emerge, & how will profit be used?
- Is project worth carrying out if potential profit is small?

Borrowing

- Can the necessary funds be borrowed when required?
- Are time limits or other restrictions imposed on borrowing? What ROI?
- Any security over other assets?

Resources

Are non-cach resources available? (People, expertise, equipment,. . .)

Risk

- Financial risks in going ahead (and in doing nothing
- Possibility of project making an unacceptably large loss. Reliability of suppliers (timing & cost)
- How certain about appropriate risk discount rate?

Economics
General economic outlook. Will interest rates rise or fall?

Cost vs Benefit
Is project worth doing?

Indirect benefit
New skills